Equity Forwards and Futures Valuation

We review the equity forward and futures pricing models. Consider an index level, \( I \), at a future time, \( T \). With respect to \( I \), we calculate 1) the forward price, 2) the futures price, and 3) delta.

We price forwards with respect to the following assumptions for dividends:

- Discrete dividend payments: we use the formula
  \[
  \frac{1}{Df_T} \left( I_0 - \sum_i Df_{T_i} \times \text{weight}_i \times d_i \right),
  \]
  where,
  
  - \( I_0 \) is the initial index level,
  - \( T \) is the time to maturity,
  - \( T_i \) is the time to the \( i^{th} \) dividend payment time,
  - \( Df_T \) is the discount factor to maturity \( T \),
  - \( Df_{T_i} \) is the discount factor to the \( i^{th} \) dividend payment time,
  - \( d_i \) is the dividend amount for the \( i^{th} \) dividend payment time,
  - \( \text{weight}_i \) is the weight of the dividend (index constituent) within the index.

Here, the term \( \sum Df \times \text{weight} \times d \) is the present value of index weighted dividend payments to maturity. We must input index weighted dividends, in particular, the
quantity \( weight_i \times d_i \) for each dividend payment time. These weighted dividends inputs could not be verified by RCM.

- Continuous dividend yield: we use the formula
  \[
  e^{-\delta T} \frac{I_0}{Df_T},
  \]
  where the input \( \delta \) is the continuously compounded dividend yield of the index.

The input \( \delta \) must be consistent with dividend yields of the index constituents. For example, if we assume that

\[
dS^k = (r - \delta^k) S^k dt + \sigma^k S^k W^k
\]

then the following relationship must hold

\[
\delta = -\frac{1}{T} \ln \left[ \sum_k \frac{weight_k \times S^k}{I} \exp(-\delta^k T) \right].
\]

We assume **deterministic interest rates** for pricing equity futures. Under these assumptions, forward and futures prices are equal (see [https://finpricing.com/lib/EqAsian.html](https://finpricing.com/lib/EqAsian.html))

For both forwards and futures, we compute delta with the following formulas

- Discrete dividend payments:
  \[
  \frac{\partial}{\partial I} \left( \frac{1}{Df_T} \left[ \left( I - \sum_i Df_{Ti} \times weight_i \times d_i \right) \right] \right) = \frac{1}{Df_T},
  \]

- Continuous dividend yield:
In general, interest rates are stochastic and index futures prices will depend on each index constituent, and not the aggregate spot index level; therefore, no formal delta exists for futures on index levels. Here, we compute a directional delta, with respect to the index’s constituent stock levels.

We compute benchmark forward prices for deterministic discrete dividend payments and constant continuous dividend yields. In particular, we used the following formulas

- Discrete dividend payments:

\[
\frac{1}{Df_T} \left( I_0 - \sum_i Df_T \times \text{weight}_i \times d_i \right),
\]

\[
\sum_i \frac{w^i S_0^i}{Df_T} e^{-\delta T}.
\]

- Continuous dividend yield:

The delta of a forward contract is then given by \(1/Df_T\) for discrete dividends, and by \(e^{-\delta T}/Df_T\) for a continuous dividend yield.

In the case of continuous dividend yield, we made the following further assumptions under the risk-neutral probability measure:

- The short-interest rate follows a H.W. process, of the form
\[ dr = (\theta - ar)dt + \sigma, dW, \]

Where

- \( a \) is a constant mean reversion parameter,
- \( \sigma \) is a constant instantaneous volatility parameter,
- \( \theta \) is chosen to match the initial term structure of risk-free rates,
- \( W \) is a standard Brownian motion.

- Each stock-price in the index follows geometric Brownian motion with drift, of the form

\[ dS^i = S^i(r-\delta)dt + S^i\sigma_i dB^i, \]

where \( B \) is a standard Brownian motion. Here \( B \) and \( W \) have a constant instantaneous correlation coefficient, \( \rho \).

Let \( f \) denote the futures price of the index; then

\[ f = \sum_i w^i F^i \exp \left[ \sigma_i^2 \left( -\frac{e^{-2aT}}{2a^2} + \frac{2e^{-aT}}{a} - \frac{3}{2a^3} + \frac{T}{a^2} \right) + \rho \sigma_i \sigma \left( \frac{T}{a} + \frac{e^{-aT}}{a^2} - \frac{1}{a^2} \right) \right], \]

where \( F \) is the forward price of the index’s \( i^{th} \) constituent.

In the case of discrete dividend, we made the following further assumptions under the risk-neutral probability measure:

- The short-interest rate follows a H.W. process of the same form as for a continuous dividend yield.
• Each stock price in the index satisfies

\[ S^i = H^i + pv^i(T) \]

where

\[ dH^i = rH^i \, dt + \sigma_i H^i \, dB^i, \quad H_0^i = S_0^i - pv_0^i(T) \]

Here,

• \( pv^i(T) \) is the present value of dividend payments to maturity \( T \),
• \( B^i \) is a standard Brownian motion,
• \( B^i \) and \( W^r \) have a constant instantaneous correlation coefficient \( \rho^i \).

Let \( f \) denote the futures price of the index; then

\[ f = \sum_i w^i F^i \exp \left[ \sigma_i^2 \left( -\frac{e^{-2\alpha T}}{2\alpha^3} + \frac{2e^{-\alpha T}}{\alpha^3} - \frac{3}{2\alpha^3} + \frac{T}{\alpha^2} \right) + \rho^i \sigma_i \left( \frac{T}{\alpha} + \frac{e^{-\alpha T}}{\alpha^2} - \frac{1}{\alpha^2} \right) \right], \]

where \( F^i \) is the forward price of the index’s \( i^{th} \) constituent.