Default Put Protection Derivative Analytics

We present a model for pricing a credit derivative product where party A has sold default put protection on a Euro denominated bond. Specifically, upon bond issuer default, party A must pay to party B a notional amount of 10 million USD (excluding accrued interest). In exchange, party B must pay a fixed rate to party A, on a quarterly basis, also based on this notional amount.

The pricing model is based on the Multiple Category Credit (MCC) model. Here we examine the model’s application towards the sale of default put protection.

We consider a swap specified by

- maturity, November 29, 2004,
- fixed rate payer, party B,
- floating rate payer, party A.

Fixed rate payments are payable by party B to party A. These payments are specified by

- notional, 10 million USD,
- quarterly payment dates, last business day of February, May 29, August 29, and November 29, beginning May 29, 2000,
- fixed rate, paid in arrears, .425% per annum.

The floating rate payable by party A to party B is specified by
• notional, 10 million USD (excluding accrued interest),
• condition to payment, with respect to reference bond issuer,
  - bankruptcy,
  - cross acceleration,
  - failure to pay,
  - repudiation,
  - restructuring.

Here, upon reference bond issuer default, party A acquires the equivalent of 10 million USD in reference bond Notes.

**Reference Bond Characteristics**

- **issuer**, Vodafone Airtouch Plc,
- **maturity**, October 27, 2006,
- **semi-annual coupon**, 5.75%,
- **original issue amount**, 1,500 million Euro.

Let the random variable $\tau$ denote the default time for the reference bond issuer. We assume that

$$Q(\tau > T) = e^{-\int_0^T h}$$

is the risk-neutral probability that the bond issuer survives to a time $T$. Here $h$ is a positive and piecewise constant function, which we refer to as a *hazard rate*. The hazard rate is the instantaneous default probability provided that there has been no prior default.
**Hazard Rate Calibration**

The hazard rate is calibrated from a set of prices for USD denominated corporate bonds, which are of the same credit class as the reference bond issuer, Vodafone Airtouch Plc. Since a corporate bond may not trade in a liquid market, however, its price in a liquid market is derived from a related, liquid asset swap, where the bond’s coupons are exchanged for USD Libor plus a spread.

Let the bonds used for calibration be arranged into a list of ascending maturity. We assume that the hazard rate is constant over the interval of time between each consecutive pair of maturities in this list. The hazard rate is calibrated by sequentially matching the price of each bond in the list against the model’s corresponding implied price.

We note that a more accurate distribution for the default time, however, may be obtained by calibrating the hazard rate against prices for asset swaps on related Euro denominated bonds of the same credit class as the reference bond issuer. We show that, if the default time and foreign exchange rate into USD are assumed to be independent, then the default time has the identical distribution under the USD risk-neutral probability measure as that under the Euro risk-neutral probability measure.

Let

- $X$ denote the default put strike level, excluding accrued interest,
- $C$ denote the reference bond’s coupon, expressed as a percentage, payable at time $t$,
- $R$ denote the reference bond issuer’s recovery rate, and
- $T_A$ and $T_B$ respectively denote the swap start and ending times.

Furthermore let
\[
\gamma(\tau) = \sum \begin{cases} 
\frac{C}{100} \left( \frac{\tau - t_{i-1}}{t_i - t_{i-1}} \right), & \text{if } t_{i-1} \leq \tau < t_i, \\
0, & \text{otherwise},
\end{cases}
\]

\[
denote accrued interest at default time \tau.
\]

The payoff to party B at default time \( \tau \), assuming that party A is credit riskless, is then given by

\[
\begin{cases} 
X(1 - R)(1 + \gamma(\tau)), & \text{if } T_A \leq \tau < T_B, \\
0, & \text{otherwise}.
\end{cases}
\]

This payoff has the USD value

\[
X (1 - R) \int_{T_A}^{T_B} (1 + \gamma(t)) B(t) f(t) dt
\]

(3.2.1)

where

- \( f(t) \) is the density function for the default time \( \tau \), under the USD risk-neutral probability measure,
- \( B(t) \) is the price, at time zero, of a riskless zero coupon bond with face value, 1 USD, and maturity, \( t \).

Let

- \( A \) denote the fixed rate payable by party B, expressed as an annualized percentage,
- \( t \) denote a fixed rate payment time,
- \( X \) be a 10 million USD notional amount.
Then the fee payable by party B at time $t$ is given by

$$\sum_{t} \begin{cases} X \frac{A}{100} \Delta t, & \text{if } \tau > t, \\ 0, & \text{otherwise,} \end{cases}$$

where $\Delta t$ is the time interval between consecutive payment dates. This fee has value

$$X \sum_{t} \frac{A}{100} (\Delta t) e^{-\frac{1}{h} B(t)}.$$  \hspace{1cm} (3.3.1)

The swap’s price from party A’s perspective is given by the difference in the fee value from the put-protection value, that is, Formula (3.3.1) less Formula (3.2.1).

**Credit Risky Counter Party**

In valuing the fee and default put components of the deal above, we assume that the counter party’s credit rating is of the same level, or better, than that of party A. The pricing model reflects this by assuming that neither party A nor party B will default during the tenor of the swap.

For more conservative pricing, however, we suggest modeling the fee payable by party B to party A as

$$\begin{cases} X \Delta t, & \text{if } \tau_v > t \text{ and } \tau_{ML} > t, \\ X \Delta t R, & \text{if } \tau_v > t \text{ and } \tau_{ML} \leq t, \\ 0, & \text{otherwise,} \end{cases}$$

Where
• $\tau_V$ is the default-time for Vodafone Airtouch Plc,
• $\tau_{ML}$ is the default-time for party B,
• $R$ is the recovery rate for party B.

Here we are explicitly treating the possibility that party B may default on this periodic payment.

Here the pricing formula incorporates the joint density for party B’s and bond issuer’s respective default-times.

**MCC Curve Routines**

We solve integrals where the integrand depends on the discount factor curve over the swap’s tenor. These integrals are computed numerically based on a type of Gaussian integration. A sufficient condition for the method’s convergence is continuity of the integrand over the region of integration. We note, however, that the discount factor curve (see [https://finpricing.com/lib/IrBasisCurve.html](https://finpricing.com/lib/IrBasisCurve.html)) is a piecewise continuous function (i.e., constant over an interval of time corresponding to a day). Sufficient conditions for convergence of the integration method are then violated.

Here we calibrate a constant hazard rate against various USD denominated corporate bonds, which are of the same credit class as the Vodafone Airtouch Plc reference bond. The bond specifications were supplied to Global Analytics from a Vodafone credit curve calculation spreadsheet, which reflects actual deal parameters. The underlying discount factor curve reflects USD Libor rates. In Table 5.1 we tabulate the bond specifications, while in Table 5.2 we compare the model and corresponding benchmark hazard rates.
<table>
<thead>
<tr>
<th>Maturity (yr.)</th>
<th>Hazard Rate ($10^{-3}$)</th>
<th>Benchmark Hazard Rate ($10^{-3}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>8.867</td>
<td>8.879</td>
</tr>
<tr>
<td>2</td>
<td>8.850</td>
<td>8.849</td>
</tr>
<tr>
<td>3</td>
<td>8.842</td>
<td>8.841</td>
</tr>
<tr>
<td>4</td>
<td>8.801</td>
<td>8.801</td>
</tr>
<tr>
<td>5</td>
<td>8.767</td>
<td>8.766</td>
</tr>
<tr>
<td>7</td>
<td>8.730</td>
<td>8.730</td>
</tr>
</tbody>
</table>

**Table 5.1.** Corporate bond specifications for hazard rate calibration.

**Table 5.2.** Comparison of hazard rate calibration from bonds of various maturities.

From Table 5.2, the relative difference of the hazard rate from the benchmark is on the order of .001% for bonds of maturity greater than or equal to two years. For the bond with one year maturity, the relative difference is approximately .1%, however; this difference may improve if a more stringent tolerance is employed for MCC’s quadrature...
and root finding algorithms. In summary, our testing shows that the hazard rate appears to be sufficiently accurately calculated.

**Hazard Rate Under Change of Measure**

Let

- \( r^{Eur}(t) \) denote the Euro short term interest rate,
- \( \tau \) denote the default-time for a Euro denominated bond,
- \( B^{Eur}(t,T) \) denote the price, at time \( t \), of a zero-coupon bond with face value, 1 Euro, and maturity, \( T \),
- \( P^{Eur} \) and \( E^{Eur} \) respectively denote the Euro risk-neutral probability measure and expectation under this measure,
- \( r^{Usd}(t) \) denote the USD short term interest rate,
- \( P^{Usd} \) and \( E^{Usd} \) respectively denote the USD risk-neutral probability measure and expectation under this measure,
- \( Q_t \) denote the spot exchange rate from Euro into USD at time \( t \),
- \( B^{Usd}(t,T) \) denote the price, at time \( t \), of a zero-coupon bond with face value, 1 USD, and maturity, \( T \).

Furthermore, assume that the respective short term interest rates, \( r^{Eur}(t) \) and \( r^{Usd}(t) \), are both deterministic functions of time. Then
\[ E^{Eur} \begin{pmatrix} \frac{1}{T} \int_{0}^{T} e^{1_{t>T}(s) ds} \\ e^{0} \end{pmatrix} = B^{Eur}(0,T)E^{Eur}(1_{t>T}), \]
\[ = B^{Eur}(0,T)P^{Eur}(\tau > T), \]
\[ = \frac{1}{Q_0} E^{Usd} \begin{pmatrix} \frac{1}{T} \int_{0}^{T} Q_T(s) ds \\ e^{0} \end{pmatrix}, \]
\[ = \frac{1}{Q_0} E^{Usd}(1_{t>T})E^{Usd} \begin{pmatrix} Q_T(s) ds \\ e^{0} \end{pmatrix}, \] assuming that \( \tau \) and \( Q_T \) are independent,
\[ = \frac{1}{Q_0} P^{Usd}(\tau > T)Q_0B^{Eur}(0,T). \]
\[ = P^{Usd}(\tau > T)B^{Eur}(0,T), \]
\[ \Rightarrow P^{Eur}(\tau > T) = P^{Usd}(\tau > T), \]
\[ \Rightarrow P^{Eur}(\tau \leq T) = P^{Usd}(\tau \leq T). \]

Assume that, for \( t > 0 \), \( \tau \) and \( Q \) are independent. Then, for \( T > 0 \),
\[ P^{Eur}(\tau \leq T) = P^{Usd}(\tau \leq T). \]

Assume also that
\[ P^{Eur}(\tau \leq T) = 1 - e^{-\int_{0}^{T} h(s) ds}, \]
where \( h(s) \) is a piecewise constant function. Then
\[ \frac{d}{dT} P^{Eur}(\tau \leq T) = h(T)e^{-\int_{0}^{T} h(s) ds}, \]
\[ = \frac{d}{dT} P^{Usd}(\tau \leq T). \]
at points, \( T \), where \( h(T) \) is continuous.

**Default Put Payoff**

Let

- \( T \) denote the reference bond’s maturity,
- \( C_i \) and \( t_i \), for \( i = 1, \ldots, N \) where \( 0 = t_0 < t_1 < \ldots < t_N = T \), be the reference bond’s coupon payable at time \( t_i \),
- \( \gamma(\tau) = \sum_{i=1}^{N} C_i \left( \frac{\tau - t_{i-1}}{t_i - t_{i-1}} \right) 1_{t_{i-1} < \tau \leq t_i} \) be the accrued interest at default time \( \tau \),
- \( X \) be the default put strike level (excluding accrued interest),
- \( R \) denote the reference bond issuer’s recovery rate,
- \( T_A \) and \( T_B \), where \( 0 < T_A < T_B < T \), respectively denote the swap start and ending times.

The payoff at default time \( \tau \) is then given by

\[
X \max\left( (1 - R)(1 + \gamma(\tau)), 0 \right) 1_{T_A < \tau \leq T_B} = X (1 - R)(1 + \gamma(\tau)) 1_{T_A < \tau \leq T_B}.
\]

This payoff has the value

\[
E\left( \frac{X (1 - R)(1 + \gamma(\tau)) 1_{T_A < \tau \leq T_B}}{\beta(\tau)} \right) = X (1 - R) \int_{T_A}^{T_B} (1 + \gamma(t)) B(0, t) f(t) dt
\]

Where

- \( E \) denotes the USD risk-neutral probability measure,
- \( f(t) \) is the probability density function for the default time \( \tau \),
• $B(0, t)$ is the price, at time zero, of a riskless zero coupon bond with face value, 1 USD, and maturity, $t$.

**Fixed Rate Fee**

Let

• $A$ denote an fixed rate payable by party B, expressed as an annualized percentage,
• $\hat{t}_i$, for $i = 1, \ldots, M$ where $T_A < \hat{t}_1 \leq \hat{t}_M = T_B$, denote a fixed rate payment time,
• $X$ denote a notional amount.

Then the fee payable by party B is given by

$$\sum_{i=1}^{M} \frac{A}{100} \Delta \hat{t}_i 1_{\tau > \hat{t}_i}$$

where $\Delta \hat{t}_i = \hat{t}_i - \hat{t}_{i-1}$ (here we set $\hat{t}_0 = T_A$). This fee has value

$$E \left( \sum_{i=1}^{M} \frac{A}{100} \Delta \hat{t}_i 1_{\tau > \hat{t}_i} \beta(\hat{t}_i) \right) = \sum_{i=1}^{M} \frac{A}{100} \Delta \hat{t}_i Q(\tau > \hat{t}_i) B(0, \hat{t}_i)$$

$$= \sum_{i=1}^{M} \frac{A}{100} \left( \Delta \hat{t}_i \right) e^{-\int_{0}^{\hat{t}_i} h(s) ds} B(0, \hat{t}_i)$$

**Credit Risky Counter Party**

Consider the sale of default-put protection to a risky counter party. Let $\tau_c$ and $\tau_B$ respectively denote the default time for the counter party and reference bond issuer. Assume that we receive at time $t_i$, for $i = 1, \ldots, M$, 


\[ C_i 1_{t_B > t_i} \left( R_c 1_{t_C \leq t_i} + 1_{t_C > t_i} \right) \]  

(A.1)

Where

- \( R_c \) is the counter party recovery rate, and
- \( C_i \) is a coupon.

Observe that (A.1) is equivalent to

\[
C \left( 1 - 1_{t_B \leq t_i} \right) \left( R_c 1_{t_C \leq t_i} + 1 - 1_{t_C > t_i} \right) = C_i \left( R_c 1_{t_C \leq t_i} + 1 - 1_{t_C > t_i} \right) - C_i 1_{t_B \leq t_i} \left( R_c 1_{t_C \leq t_i} + 1 - 1_{t_C > t_i} \right),
\]

\[
= C_i \left[ 1 + 1_{t_C \leq t_i} \left( R_c - 1 \right) - 1_{t_B \leq t_i} \right] - 1_{t_B \leq t_i} \left( R_c - 1 \right),
\]

\[
= C_i \left[ 1 - 1_{t_B \leq t_i} + \left( R_c - 1 \right) \left( 1_{t_C \leq t_i} - 1_{t_B \leq t_i} \right) \right].
\]

The payoff above has value

\[
E \left( \frac{C \left( 1 - 1_{t_B \leq t_i} \right) \left( R_c - 1 \right) \left( 1 - F_B \left( t_i \right) \right) \left( F_C \left( t_i \right) - F \left( t_i, t_i \right) \right) }{\beta_i} \right)
\]

\[
= C_i d \left( t_i \right) \left[ 1 - F_B \left( t_i \right) \right] + \left( R_c - 1 \right) \left( F_C \left( t_i \right) - F \left( t_i, t_i \right) \right),
\]

\[
= C_i d \left( t_i \right) \left[ 1 - F_B \left( t_i \right) \right] + \left( R_c - 1 \right) F_C \left( t_i \right) \left[ 1 - F_B \left( t_i \right) \left( 1 - \rho + \frac{\rho}{G \left( t_i \right)} \right) \right],
\]

where

- \( E \) is the USD risk-neutral probability measure,
- \( d(t) \) is the price of a riskless zero coupon bond with maturity, \( t \), and face value, 1 USD,
- \( F_B(t) = P(\tau_B \leq t) = 1 - e^{-\int_0^t b_B(s) ds} \),
- \( F_C(t) = P(\tau_C \leq t) = 1 - e^{-\int_0^t b_C(s) ds} \),
• $h_b(t)$ and $h_c(t)$ are the hazard rate functions corresponding to the bond issuer and counter party default time,

• $\rho$, where $0 \leq \rho < 1$, is an exogenous parameter

(Here \( P \) denotes the USD risk-neutral probability measure).

In the above,

$$F(u,v) = \frac{F_b(u)F_c(v)}{G(u)G(v)} \left( (1 - \rho)G(u)G(v) + \rho \min(G(u),G(v)) \right),$$

Where

• $G(t) = \int_0^t g(s) ds$,

• $g(t) = \lambda(t) e^{-\lambda(t)ds}$,

• $\lambda(t) = \max(h_b(t),h_c(t))$.

**Benchmark Hazard Rate Calibration**

Consider calibrating the hazard rate to a single corporate bond. In this case, the hazard rate has a constant value, $h$.

Let $D_i$, for $i = 1,...,N$, be an increasing sequence of dates such that the corporate bond’s $i^{th}$ coupon, $C_i$, is paid at $D_i$ (here we let $D_0$ denote the valuation date). Furthermore, for $i = 1,...,N$, let

• $t_i = \frac{D_i - D_0}{365}$, be a time that corresponds to $D_i$ (here we set $t_0 = 0$),

• $n_i = 1 + D_i - D_{i-1}$.
Also, for \( i = 1, \ldots, N \), let

- \( \tau_i^0 = t_{i-1} \),
- \( \tau_i^j = t_{i-1} + \frac{j - 1}{2} \), for \( j = 1, \ldots, n_i - 1 \),
- \( \tau_{n_i}^i = t_i \)

(here we set \( t_0 = 0 \)). Finally, for \( i = 1, \ldots, N \), let

\[
d_j^i = df(D_i + j - 1),
\]

for \( j = 1, \ldots, n_i \), be the discount factor corresponding to the date that is \( j - 1 \) days in the future from \( D_i \).

The bond’s price then equals

\[
\zeta(h) + \sum_{i=1}^N C_i d_i^i e^{-h_i} + d_{n_i}^N e^{-h_N},
\]

where \( \zeta(h) \) equals
\[
R \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left(1 + C_i \frac{t-t_{i-1}}{t_i-t_{i-1}} \right) h e^{\lambda_s \beta} \sum_{j=1}^{n} d_j \frac{1}{t_j-t_{i-1}} \, dt_i \sum_{j=1}^{n} d_j \frac{1}{t_j-t_{i-1}} \, dt,
\]

\[
= R \sum_{i=1}^{n} \sum_{j=1}^{n} \int_{t_{i-1}}^{t_i} \left(1 + C_i \frac{t-t_{i-1}}{t_i-t_{i-1}} \right) d_j h e^{\lambda_s \beta} \, dt_i.
\]

\[
= R \sum_{i=1}^{n} \sum_{j=1}^{n} \left( d_j h \int_{t_{i-1}}^{t_i} e^{\lambda_s \beta} \, dt_i + \frac{C_i d_i h}{t_i-t_{i-1}} \int_{t_{i-1}}^{t_i} (t-t_{i-1}) e^{\lambda_s \beta} \, dt_i \right).
\]

\[
= R \sum_{i=1}^{n} \sum_{j=1}^{n} \left( d_j h \int_{t_{i-1}}^{t_i} e^{\lambda_s \beta} \, dt_i + \frac{C_i d_i h}{t_i-t_{i-1}} \left[ \int_{t_{i-1}}^{t_i} \frac{e^{\lambda_s \beta}}{h} \, dt_i + \left( \frac{1}{h} - t_{i-1} \right) \int_{t_{i-1}}^{t_i} e^{\lambda_s \beta} \, dt_i \right] \right).
\]

\[
= R \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \frac{C_i d_j}{t_i-t_{i-1}} \left[ t_i e^{\lambda_s \beta t_i} - t_i e^{\lambda_s \beta t_{i-1}} \right] + d_j \left[ 1 + \frac{C_i}{t_i-t_{i-1}} \left( \frac{1}{h} - t_{i-1} \right) \right] \right) e^{\lambda_s \beta t_{i-1}}.
\]