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EVOLUTIONARY MATHEMATICS AND SCIENCE FOR ULTIMATE GENERALIZATION OF LAH NUMBERS/(BINOMIAL COEFFICIENTS): SUMS/(ALTERNATE SUMS) OF ORTHOGONAL PRODUCTS OF STIRLING NUMBERS

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TABLE OF CONTENTS

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ABSTRACT

We first introduce Stirling and Lah numbers via recursion and express Lah numbers and binomial coefficients as sums and alternate sums of orthogonal products of Stirling numbers of both kinds, respectively. After pointing out that Fibonacci numbers are nothing but upward diagonal sums of Pascal triangle, we generalize the triangular arrays in question from the natural sequence based to arithmetically progressive sequences based and call their upward diagonal sum Fibonacci values. After looking at more triangular arrays based on other sequences such as binomial coefficients and Fibonacci numbers, we eventually conclude that such construction of triangular arrays works with any underlying sequence base.

Keywords: Binomial coefficient, Stirling number, Lah number, Sum, Alternate sum, Orthogonal product, Natural sequence, Arithmetically progressive sequence, Recursion, Fibonacci number, upward diagonal, Fibonacci value, q-Gaussian coefficient.

NOMENCLATURE

$\binom{n}{k}$	binomial coefficient
Σ	sum
$\prod$	product
$(i)_1^{\infty}$	the natural sequence
$\left]{n \atop k}\right[$	Lah number
$\left[{n \atop k}\right]$	Stirling number of the first kind
$\left\{{n \atop k}\right\}$	Stirling number of the second kind
$(a+(i-1)d)_1^{\infty}$	arithmetically progressive sequence
$\left[{n \atop k}\right]_{a;d}$	Stirling triangle of the first kind for $(a+(i-1)d)_1^{\infty}$
$\left\{{n \atop k}\right\}_{a;d}$	Stirling triangle of the second kind for $(a+(i-1)d)_1^{\infty}$
$\left[{n \atop k}\right]_{(a_i)_1^{\infty}}$	Stirling triangle of the first kind for $(a_i)_1^{\infty}$
$\left\{{n \atop k}\right\}_{(a_i)_1^{\infty}}$	Stirling triangle of the second kind for $(a_i)_1^{\infty}$
$\left]{n \atop k}\right[_{(a_i)_1^{\infty}}$	Lah number for $(a_i)_1^{\infty}$
$\binom{n}{k}_{(a_i)_1^{\infty}}$	binomial coefficient for $(a_i)_1^{\infty}$

1. INTRODUCTION

It is well-known that Stirling numbers of the first kind $\left[{n \atop k}\right]$ can be tabulated recursive via

$$\left[{n \atop k}\right] = \left[{n-1 \atop k-1}\right] + (n-1)\left[{n-1 \atop k}\right] , \qquad \text{Eq. 1}$$

with the initial value $\left[{1 \atop 1}\right] = 1$, as follows.

n\k	1	2	3	4	5	6	7	8	9	10
1	1									
2	1	1								
3	2	3	1							
4	6	11	6	1						
5	24	50	35	10	1					
6	120	274	225	85	15	1				
7	720	1764	1624	735	175	21	1			
8	5040	13068	13132	6769	1960	322	28	1		
9	40320	109584	118124	67284	22449	4536	546	36	1	
10	362880	1026576	1172700	723680	269325	63273	9450	870	45	1

Table 1: Table for Stirling numbers of the first kind

Likewise, Stirling numbers of the second kind $\left\{{n \atop k}\right\}$ can be tabulated recursive via

$$\left\{{n \atop k}\right\} = \left\{{n-1 \atop k-1}\right\} + k\left\{{n-1 \atop k}\right\} , \qquad \text{Eq. 2}$$

with the initial value $\left\{{1 \atop 1}\right\} = 1$, as follows.

7

n\k	1	2	3	4	5	6	7	8	9	10
1	1									
2	1	1								
3	1	3	1							
4	1	7	6	1						
5	1	15	25	10	1					
6	1	31	90	65	15	1				
7	1	63	301	350	140	21	1			
8	1	127	966	1701	1050	266	28	1		
9	1	255	3025	7770	6951	2646	462	36	1	
10	1	511	9330	34105	42525	22827	5880	750	45	1

Table 2: Table for Stirling numbers of the second kind

On the other hand, binomial coefficients $\binom{n}{k}$ can be tabulated recursive via

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \qquad \text{Eq. 3}$$

as follows.

n\k	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	1									
2	1	2	1								
3	1	3	3	1							
4	1	4	6	4	1						
5	1	5	10	10	5	1					
6	1	6	15	20	15	6	1				
7	1	7	21	35	35	21	7	1			
8	1	8	28	56	70	56	28	8	1		
9	1	9	36	84	126	126	84	36	9	1	
10	1	10	45	120	210	252	210	120	45	10	1

Table 3. Pascal Triangle

It is quite amazing that Table 3 can be reproduced from Tables 1 and 2 via

$$\binom{n}{k} = \sum_{j=k}^{n} (-1)^{j-k} \genfrac{[}{]}{0pt}{}{j}{k} \genfrac{\{}{\}}{0pt}{}{n+1}{j+1} , \qquad \text{Eq. 4}$$

which was obtained in (2). For example,

$$\binom{5}{2} = \genfrac{[}{]}{0pt}{}{2}{2}\genfrac{\{}{\}}{0pt}{}{6}{3} - \genfrac{[}{]}{0pt}{}{3}{2}\genfrac{\{}{\}}{0pt}{}{6}{4} + \genfrac{[}{]}{0pt}{}{4}{2}\genfrac{\{}{\}}{0pt}{}{6}{5} - \genfrac{[}{]}{0pt}{}{5}{2}\genfrac{\{}{\}}{0pt}{}{6}{6} = (1)(90)-(3)(65)+(11)(15)-(50)(1) = 10,$$

$$\binom{5}{3} = \genfrac{[}{]}{0pt}{}{3}{3}\genfrac{\{}{\}}{0pt}{}{6}{4} - \genfrac{[}{]}{0pt}{}{4}{3}\genfrac{\{}{\}}{0pt}{}{6}{5} + \genfrac{[}{]}{0pt}{}{5}{3}\genfrac{\{}{\}}{0pt}{}{6}{6} = (1)(65)-(6)(15)+(35)(1) = 10,$$

$$\binom{6}{3} = \genfrac{[}{]}{0pt}{}{3}{3}\genfrac{\{}{\}}{0pt}{}{7}{4} - \genfrac{[}{]}{0pt}{}{4}{3}\genfrac{\{}{\}}{0pt}{}{7}{5} + \genfrac{[}{]}{0pt}{}{5}{3}\genfrac{\{}{\}}{0pt}{}{7}{6} - \genfrac{[}{]}{0pt}{}{6}{3}\genfrac{\{}{\}}{0pt}{}{7}{7} = (1)(350)-(6)(140)+(35)(21)-(225)(1) = 20,$$

which also verify Eq. 3.

Now let us prove Eq. 4 by mathematical induction. We shall only look at the case for $n = 5$ and $k = 3$, since the general case is similar. We can use Eqs. 2 and 3 to show the inductive step:

$$\binom{6}{3} = \left(\genfrac{[}{]}{0pt}{}{3}{3}\genfrac{\{}{\}}{0pt}{}{6}{4} - \genfrac{[}{]}{0pt}{}{4}{3}\genfrac{\{}{\}}{0pt}{}{6}{5} + \genfrac{[}{]}{0pt}{}{5}{3}\genfrac{\{}{\}}{0pt}{}{6}{6} \right) + \left(\genfrac{[}{]}{0pt}{}{3}{3}\genfrac{\{}{\}}{0pt}{}{6}{3} - \genfrac{[}{]}{0pt}{}{4}{3}\genfrac{\{}{\}}{0pt}{}{6}{4} + \genfrac{[}{]}{0pt}{}{5}{3}\genfrac{\{}{\}}{0pt}{}{6}{5} - \genfrac{[}{]}{0pt}{}{6}{3}\genfrac{\{}{\}}{0pt}{}{6}{6} \right)$$

$$= \genfrac{[}{]}{0pt}{}{2}{2}\genfrac{\{}{\}}{0pt}{}{6}{3} - \left(\genfrac{[}{]}{0pt}{}{4}{3} - 4\genfrac{[}{]}{0pt}{}{3}{3} \right)\genfrac{\{}{\}}{0pt}{}{6}{4} + \left(\genfrac{[}{]}{0pt}{}{5}{3} - 5\genfrac{[}{]}{0pt}{}{4}{3} \right)\genfrac{\{}{\}}{0pt}{}{6}{5} - \left(\genfrac{[}{]}{0pt}{}{6}{3} - 6\genfrac{[}{]}{0pt}{}{5}{3} \right)\genfrac{\{}{\}}{0pt}{}{6}{6}$$

$$= \left(4\genfrac{\{}{\}}{0pt}{}{6}{4} + \genfrac{\{}{\}}{0pt}{}{6}{3} \right) - \genfrac{[}{]}{0pt}{}{4}{3}\left(5\genfrac{\{}{\}}{0pt}{}{6}{5} + \genfrac{\{}{\}}{0pt}{}{6}{4} \right) + \genfrac{[}{]}{0pt}{}{5}{3}\left(6\genfrac{\{}{\}}{0pt}{}{6}{6} + \genfrac{\{}{\}}{0pt}{}{6}{5} \right) - \genfrac{[}{]}{0pt}{}{6}{3}\genfrac{\{}{\}}{0pt}{}{6}{6}$$

$$= \genfrac{[}{]}{0pt}{}{3}{3}\genfrac{\{}{\}}{0pt}{}{7}{4} - \genfrac{[}{]}{0pt}{}{4}{3}\genfrac{\{}{\}}{0pt}{}{7}{5} + \genfrac{[}{]}{0pt}{}{5}{3}\genfrac{\{}{\}}{0pt}{}{7}{6} - \genfrac{[}{]}{0pt}{}{6}{3}\genfrac{\{}{\}}{0pt}{}{7}{7}.$$

Instead of taking the alternate sum of orthogonal products of Stirling numbers of both kinds for the compressed notation $\binom{n}{k}$ as in Eq. 4, we can obtain Lah number $\left]\begin{matrix}n\\k\end{matrix}\right[$ via

$$\left]\begin{matrix}n\\k\end{matrix}\right[= \sum_{j=1}^{n} \begin{bmatrix}n\\j\end{bmatrix} \begin{Bmatrix}j\\k\end{Bmatrix} \qquad \text{Eq. 5}$$

for expanded notation $\left]\begin{matrix}n\\k\end{matrix}\right[$, whereas Lah numbers are usually tabulated by way of recursion

$$\left]\begin{matrix}n\\k\end{matrix}\right[= \left]\begin{matrix}n-1\\k-1\end{matrix}\right[+ (n+k-1)\left]\begin{matrix}n-1\\k\end{matrix}\right[\qquad \text{Eq. 6}$$

as follows.

n\k	1	2	3	4	5	6	7	8	9	10
1	1									
2	2	1								
3	6	6	1							
4	24	36	12	1						
5	120	240	120	20	1					
6	720	1800	1200	300	30	1				
7	5040	15120	12600	4200	630	42	1			
8	40320	141120	141120	58800	11760	1176	56	1		
9	362880	1451520	1693440	846720	211680	28224	2016	72	1	
10	3628800	16329600	21772800	12700800	3810240	635040	60480	3240	90	1

Table 4. Table for Lah numbers $\left]\begin{matrix}n\\k\end{matrix}\right[$

The following formula in (1), with $\left\lfloor\frac{n-1}{2}\right\rfloor$ denoting the largest integer no greater than $\frac{n-1}{2}$,

$$F_n = \sum_{k=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor} \binom{n-k-1}{k} , \qquad \text{Eq. 7}$$

is, in fact, the sum of the nth upward diagonal of Pascal triangle in Table 3.

From a broader prospective, we realize that the triangular arrays displayed in Tables 1-4 can be construed as two-legged recursive structures based on the natural sequence $(i)_1^{\infty}$, with one weighted leg.

Accordingly, let us denote them as $\begin{bmatrix} n \\ k \end{bmatrix}_{(i)_1^\infty}$, $\begin{Bmatrix} n \\ k \end{Bmatrix}_{(i)_1^\infty}$, $\begin{pmatrix} n \\ k \end{pmatrix}_{(i)_1^\infty}$, $\left]\begin{matrix} n \\ k \end{matrix}\right[_{(i)_1^\infty}$ and the nth Fibonacci value F_n as $F_n((i)_1^\infty)$.

Before going any further, let us recap with the two-legged recursive structure based on the unity sequence $(1)_1^\infty$. Readers can first verify $\begin{bmatrix} n \\ k \end{bmatrix}_{(1)_1^\infty} = \begin{Bmatrix} n \\ k \end{Bmatrix}_{(1)_1^\infty} = \begin{pmatrix} n \\ k \end{pmatrix}_{(i)_1^\infty}$, then calculate

$\begin{pmatrix} n \\ k \end{pmatrix}_{(1)_1^\infty}$ to be a triangular array with 0 entries except for the rightmost diagonal entries being 1 so that $F_{2n-1}((1)_1^\infty) = 1$ and $F_{2n}((1)_1^\infty) = 0$, finally tabulate $\left]\begin{matrix} n \\ k \end{matrix}\right[_{(1)_1^\infty}$ as follows via

$$\left]\begin{matrix} n \\ k \end{matrix}\right[_{(1)_1^\infty} = \left]\begin{matrix} n-1 \\ k-1 \end{matrix}\right[_{(1)_1^\infty} + 2\left]\begin{matrix} n-1 \\ k \end{matrix}\right[_{(1)_1^\infty} . \qquad \text{Eq. 8}$$

n\k	1	2	3	4	5	6
1	1					
2	2	1				
3	4	4	1			
4	8	12	6	1		
5	16	32	24	8	1	
6	32	80	120	40	10	1

Table 5. Table for Lah numbers $\left]\begin{matrix} n \\ k \end{matrix}\right[_{(1)_1^\infty}$

2. GENERALIZTION

For any a and d in a commutative ring, triangular arrays of Stirling numbers of both kinds based on an arithmetically progressive sequence $(a+(i-1)d)_1^\infty$ have been introduced and denoted in (3) as $\begin{bmatrix} n \\ k \end{bmatrix}_{a;d}$ with the recursive formula for the first kind

$$\begin{bmatrix} n \\ k \end{bmatrix}_{a;d} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{a;d} + [a+(n-2)d]\begin{bmatrix} n-1 \\ k \end{bmatrix}_{a;d} \qquad \text{Eq. 9}$$

and the initial value $\begin{bmatrix} n \\ n \end{bmatrix}_{a;d} = 1$ and $\begin{Bmatrix} n \\ k \end{Bmatrix}_{a;d}$ with the recursive formula for of the second kind

$$\begin{Bmatrix} n \\ k \end{Bmatrix}_{a;d} = \begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix}_{a;d} + [a+(k-1)d]\begin{Bmatrix} n-1 \\ k \end{Bmatrix}_{a;d} \qquad \text{Eq. 10}$$

and the initial value $\begin{Bmatrix} n \\ n \end{Bmatrix}_{a;d} = 1$.

To elaborate, we take $a=2$ and $d=3$. Using Eq. 9, we can tabulate $\begin{bmatrix} n \\ k \end{bmatrix}_{2;3}$ in Table 6.

$n \backslash k$	1	2	3	4	5	6	7
1	1						
2	2	1					
3	10	7	1				
4	80	66	15	1			
5	880	806	231	26	1		
6	12320	12164	4040	595	40	1	
7	209440	219108	80844	14155	1275	57	1

Table 6. Table for Stirling numbers of the first kind $\begin{bmatrix} n \\ k \end{bmatrix}_{2;3}$

Likewise, we can use Eq. 10 to tabulate $\left\{ {n \atop k} \right\}_{2;3}$ in Table 7.

$n \backslash k$	1	2	3	4	5	6	7
1	1						
2	2	1					
3	4	7	1				
4	8	39	15	1			
5	16	203	159	26	1		
6	32	1031	1475	445	40	1	
7	64	5187	12831	6370	1005	57	1

Table 7. Table for Stirling numbers of the second kind $\left\{ {n \atop k} \right\}_{2;3}$

In (2), Eq. 4 was generalized to be

$$\binom{n}{k}_{a;d} = \sum_{j=k}^{n} (-1)^{j-k} \left[{j \atop k} \right]_{a;d} \left\{ {n+1 \atop j+1} \right\}_{a;d} \qquad \text{Eq. 11}$$

and used to prove the main theorem in (3).

Furthermore, we can generalize Eqs. 5 and 6 to be

$$\left] {n \atop k} \right[_{a;d} = \sum_{j=0}^{n-k} \left[{n \atop k+j} \right]_{a;d} \left\{ {k+j \atop k} \right\}_{a;d} , \qquad \text{Eq. 12}$$

And

$$\left] {n \atop k} \right[_{a;d} = \left] {n-1 \atop k-1} \right[_{a;d} + \{[a + (n-2)d] + [a + (k-1)d]\} \left] {n-1 \atop k} \right[_{a;d} . \qquad \text{Eq. 13}$$

Taking a=2 and d=3 for example, readers can verify Eqs. 11-13 with Tables 6-8.

n\k	1	2	3	4	5	6	7
1	1						
2	4	1					
3	28	14	1				
4	280	210	30	1			
5	3640	3640	780	52	1		
6	58240	72800	20800	2080	80	1	
7	1106560	1659840	592800	79040	4560	114	1

Table 8. Table for Lah numbers $\left]\begin{matrix} n \\ k \end{matrix}\right[_{a;d}$

Finally, Eq. 7 can be generalized to be

$$F_n(a,d) = \sum_{k=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor} \binom{n-k-1}{k}_{a;d}. \qquad \text{Eq. 14}$$

We further note that the recursive formula for $\binom{n}{k}_{a;d}$ can be proved to be

$$\binom{n}{k}_{a;d} = \binom{n-1}{k-1}_{a;d} + \mathrm{d}\binom{n-1}{k}_{a;d} \qquad \text{Eq. 15}$$

with the initial values $\binom{n}{0}_{a;d} = a^n$.

Before giving the following example, we would like to point out that $\binom{n}{k}_{a;d}$ is the generalization of $\binom{n}{k}$, which is $\binom{n}{k}_{1;1}$, from (1, 1) to (a, d), rather than from the natural sequence based to arithmetically progressive sequence based. For instance, the following recursive formula is good for any sequence 2, 2+3, x, y, z, … (Could this feature be useful in Cryptography?)

Now, by taking a=2 and d=3 in Eq. 15, we can use the initial values $\binom{n}{0}_{2;3} = 2^n$ to calculate the entries of Table 9.

$n \backslash k$	0	1	2	3	4	5	6
0	1						
1	2	1					
2	4	5	1				
3	8	19	8	1			
4	16	65	43	11	1		
5	32	211	194	76	14	1	
6	64	655	793	422	118	17	1

Table 9. Table for $\binom{n}{k}_{2;3}$

Finally, we can calculate $F_n(2;3)$ from Table 9 via Eq. 14 as follows.

$F_1(2,3) = \binom{0}{0}_{2.3} = 1$, $F_2(2,3) = \binom{1}{0}_{2.3} = 2$,

$F_3(2,3) = \binom{2}{0}_{2.3} + \binom{1}{1}_{2.3} = 5$, $F_4(2,3) = \binom{3}{0}_{2.3} + \binom{2}{1}_{2.3} = 13$,

$F_5(2,3) = \binom{4}{0}_{2.3} + \binom{3}{1}_{2.3} + \binom{2}{2}_{2.3} = 36$,

$F_6(2,3) = \binom{5}{0}_{2.3} + \binom{4}{1}_{2.3} + \binom{3}{2}_{2.3} = 105, \ldots$

In stead of using Eq. 15, we can directly sum up the nth upward diagonal of Table 9 to obtain the nth Fibonacci value $F_n(2;3)$!

Next, we consider $\begin{bmatrix} n \\ k \end{bmatrix}_{\left(\binom{i+1}{2}\right)_1^{\infty}}$ and $\begin{Bmatrix} n \\ k \end{Bmatrix}_{\left(\binom{i+1}{2}\right)_1^{\infty}}$. We can use

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\left(\binom{i+1}{2}\right)_1^{\infty}} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{\left(\binom{i+1}{2}\right)_1^{\infty}} + \binom{n}{2} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{\left(\binom{i+1}{2}\right)_1^{\infty}} \qquad \text{Eq. 16}$$

to tabulate Table 10.

$n \backslash k$	1	2	3	4	5	6	7
1	1						
2	1	1					
3	3	4	1				
4	18	27	10	1			
5	180	288	127	20	1		
6	2700	4500	2193	427	35	1	
7	56700	97200	50553	11160	1162	56	1

Table 10. Table for $\begin{bmatrix} n \\ k \end{bmatrix}_{\left(\binom{i+1}{2}\right)_1^{\infty}}$

Likewise, we can use

$$\begin{Bmatrix} n \\ k \end{Bmatrix}_{\left(\binom{i+1}{2}\right)_1^{\infty}} = \begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix}_{\left(\binom{i+1}{2}\right)_1^{\infty}} + \binom{k+1}{2} \begin{Bmatrix} n-1 \\ k \end{Bmatrix}_{\left(\binom{i+1}{2}\right)_1^{\infty}} \qquad \text{Eq. 17}$$

to tabulate Table 11.

$n \backslash k$	1	2	3	4	5	6	7
1	1						
2	1	1					
3	1	7	1				
4	1	43	17	1			
5	1	259	213	32	1		
6	1	1555	2389	693	53	1	
7	1	9331	25445	12784	1806	81	1

Table 11. Table for $\begin{Bmatrix} n \\ k \end{Bmatrix}_{\left(\binom{i+1}{2}\right)_1^{\infty}}$

Similar to Eq. 12, we define $\left]{n \atop k}\right[_{\left(\binom{i+1}{2}\right)_1^\infty}$ to be

$$\left]{n \atop k}\right[_{\left(\binom{i+1}{2}\right)_1^\infty} = \sum_{j=0}^{n-k} \begin{bmatrix} n \\ k+j \end{bmatrix}_{\left(\binom{i+1}{2}\right)_1^\infty} \begin{Bmatrix} k+j \\ k \end{Bmatrix}_{\left(\binom{i+1}{2}\right)_1^\infty} \qquad \text{Eq. 18}$$

and expect the recursive formula for which to be comparable to Eq. 13:

$$\left]{n \atop k}\right[_{\left(\binom{i+1}{2}\right)_1^\infty} = \left]{n-1 \atop k-1}\right[_{\left(\binom{i+1}{2}\right)_1^\infty} + \left[\binom{n}{2} + \binom{k+1}{2}\right] \left]{n-1 \atop k}\right[_{\left(\binom{i+1}{2}\right)_1^\infty} . \qquad \text{Eq. 19}$$

From Tables 10 and 11, we can use Eq. 18 to come up with Table 12 below.

n\k	1	2	3	4	5	6	7
1	1						
2	2	1					
3	8	11	1				
4	56	140	27	1			
5	616	2296	680	52	1		
6	9856	48832	19296	2240	88	1	
7	216832	1328320	647008	99936	5936	137	1

Table 12. Table for Lah numbers $\left]{n \atop k}\right[_{\left(\binom{i+1}{2}\right)_1^\infty}$

In fact, we can use Table 12 to verify Eq. 19. Finally, we use

$$\binom{n}{k}_{\left(\binom{i+1}{2}\right)_1^\infty} = \sum_{j=k}^{n} (-1)^{j-k} \begin{bmatrix} j \\ k \end{bmatrix}_{\left(\binom{i+1}{2}\right)_1^\infty} \begin{Bmatrix} n+1 \\ j+1 \end{Bmatrix}_{\left(\binom{i+1}{2}\right)_1^\infty} \qquad \text{Eq. 20}$$

to come up with Table 13.

$n \setminus k$	1	2	3	4	5	6
1	1					
2	6	1				
3	29	13	1			
4	124	112	22	1		
5	471	760	290	33	1	
6	1610	4243	2818	613	46	1

Table 13. Table for $\binom{n}{k}_{\left(\binom{i+1}{2}\right)_1^\infty}$

From Table 13, we can sum up the entries of upward diagonals to obtain the first six

Fibonacci values as $F_1\left(\binom{i+1}{2}_1^\infty\right) = 1$, $F_2\left(\binom{i+1}{2}_1^\infty\right) = 8$, $F_3\left(\binom{i+1}{2}_1^\infty\right) = 30$,

$F_4\left(\binom{i+1}{2}_1^\infty\right) = 137$, $F_5\left(\binom{i+1}{2}_1^\infty\right) = 584$ and $F_6\left(\binom{i+1}{2}_1^\infty\right) = 2402$.

We further consider Stirling numbers based on $\left(q^{i-1}\right)_1^\infty$ with $q \neq 0$. From

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\left(q^{i-1}\right)_1^\infty} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{\left(q^{i-1}\right)_1^\infty} + q^{n-2} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{\left(q^{i-1}\right)_1^\infty} \qquad \text{Eq. 21}$$

and

$$\begin{Bmatrix} n \\ k \end{Bmatrix}_{\left(q^{i-1}\right)_1^\infty} = \begin{Bmatrix} n-1 \\ k = 1 \end{Bmatrix}_{\left(q^{i-1}\right)_1^\infty} + q^{k-1} \begin{Bmatrix} n-1 \\ k \end{Bmatrix}_{\left(q^{i-1}\right)_1^\infty}, \qquad \text{Eq. 22}$$

it is quite easy to come up with

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\left(q^{i-1}\right)_1^\infty} = q^{\binom{n-k}{2}} \prod_{i=1}^{k-1} \frac{1-q^{n-k+i}}{1-q^{i}} = q^{\binom{n-k}{2}} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q$$

and

$$\begin{Bmatrix} n \\ k \end{Bmatrix}_{\left(q^{i-1}\right)_1^\infty} = \prod_{i=1}^{k-1} \frac{1-q^{n-k+i}}{1-q^{i}} = \begin{bmatrix} n-1 \\ k \end{bmatrix}_q ,$$

where $\begin{bmatrix} n-1 \\ k \end{bmatrix}_q$ is known to be a q-Gaussian coefficient.

Taking q=2 for example, we can use Eq. 21 with $\begin{bmatrix}1\\1\end{bmatrix}_{(2^{i-1})_1^\infty} = 1$ to produce Table 14.

$n \setminus k$	1	2	3	4	5	6
1	1					
2	1	1				
3	2	3	1			
4	8	14	7	1		
5	64	120	70	15	1	
6	1024	1984	1240	310	31	1

Table 14. Table for Stirling numbers of the first kind $\begin{bmatrix}n\\k\end{bmatrix}_{(2^{i-1})_1^\infty}$

Likewise, we can use Eq. 22 with $\begin{Bmatrix}1\\1\end{Bmatrix}_{(2^{i-1})_1^\infty} = 1$ to produce Table 15.

$n \setminus k$	1	2	3	4	5	6
1	1					
2	1	1				
3	1	3	1			
4	1	7	7	1		
5	1	15	35	15	1	
6	1	31	155	155	31	1

Table 15. Table for Stirling numbers of the second kind $\begin{Bmatrix}n\\k\end{Bmatrix}_{(2^{i-1})_1^\infty}$

We can further use

$$\left]\begin{matrix}n\\k\end{matrix}\right[_{(2^{i-1})_1^\infty} = \sum_{j=0}^{n-k} \begin{bmatrix}n\\k+j\end{bmatrix}_{(2^{i-1})_1^\infty} \begin{Bmatrix}k+j\\k\end{Bmatrix}_{(2^{i-1})_1^\infty} \qquad \text{Eq. 23}$$

to come up with Table 16.

$n \setminus k$	1	2	3	4	5	6
1	1					
2	2	1				
3	6	6	1			
4	30	42	14	1		
5	270	450	210	30	1	
6	4590	8370	4650	930	62	1

Table 16. Table for Lah numbers $\left]\begin{matrix}n\\k\end{matrix}\right[_{(2^{i-1})_1^\infty}$

Similar to Eq. 19, we can use Table 16 to verify

$$\left]\begin{matrix} n \\ k \end{matrix}\right[_{(2^{i-1})_1^\infty} = \left]\begin{matrix} n-1 \\ k-1 \end{matrix}\right[_{(2^{i-1})_1^\infty} + [2^{n-2} + 2^{k-1}] \left]\begin{matrix} n-1 \\ k \end{matrix}\right[_{(2^{i-1})_1^\infty}, \qquad \text{Eq. 24}$$

which along with Table 16 can be readily generalized to and Table 17 below.

$$\left]\begin{matrix} n \\ k \end{matrix}\right[_{(q^{i-1})_1^\infty} = \left]\begin{matrix} n-1 \\ k-1 \end{matrix}\right[_{(q^{i-1})_1^\infty} + [q^{n-2} + q^{k-1}] \left]\begin{matrix} n-1 \\ k \end{matrix}\right[_{(q^{i-1})_1^\infty} \qquad \text{Eq. 25}$$

$n \backslash k$	1	2	3	4	5
1	1				
2	2	1			
3	2(1+q)	2(1+q)	1		
4	2(1+q)(1+q^2)	2(1+q)(1+$q + q^2$)	2(1+$q + q^2$)	1	
5	2(1+q)(1+q^2)(1+q^3)	2(1+q)(1+q+q^2)(1++$q + q^2 + q^3$)	2(1+q)(1+q^2)(1++$q + q^2$)	2(1++$q + q^2 + q^3$)	1

Table 17. Table for $\left]\begin{matrix} n \\ k \end{matrix}\right[_{(q^{i-1})_1^\infty}$

Finally, we can use

$$\binom{n}{k}_{(q^{i-1})_1^\infty} = \sum_{j=k}^{n} (-1)^{j-k} \begin{bmatrix} j \\ k \end{bmatrix}_{(q^{i-1})_1^\infty} \begin{Bmatrix} n+1 \\ j+1 \end{Bmatrix}_{(q^{i-1})_1^\infty} \qquad \text{Eq. 26}$$

to come up with Table 18 as follows.

$n \backslash k$	1	2	3	4	5
1	1				
2	q	1			
3	q	q^2	1		
4	q	q^2	q^3	1	
5	q	q^2	q^3	q^4	1

Table 18. Table for $\binom{n}{k}_{(q^{i-1})_1^\infty}$

From Table 18, we can sum up the entries of upward diagonals to obtain Fibonacci values

as $F_1\left((q^{i-1})_1^\infty\right) = 1$, $F_{2n}\left((q^{i-1})_1^\infty\right) = \sum_{j=1}^{n} q^j$, $F_{2n+1}\left((q^{i-1})_1^\infty\right) = 1 + F_{2n}\left((q^{i-1})_1^\infty\right)$.

3. CONCLUSION

To close out, let us look at the pertinent triangular arrays based on the usual Fibonacci sequence: $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, $F_6 = 8$, $F_7 = 13$, $F_8 = 21$, $F_9 = 34$, … displayed in Tables 19-23 below.

n\k	1	2	3	4	5	6	7	8	9	10
1	1									
2	1	1								
3	1	2	1							
4	2	5	4	1						
5	6	17	17	7	1					
6	30	91	102	52	12	1				
7	240	758	907	518	148	20	1			
8	3120	10094	12549	7641	2442	408	33	1		
9	66520	215094	273623	173010	58923	11010	1101	54	1	
10	2227680	7378716	9518276	6155963	2176392	433263	48444	2937	88	1

Table 19. Table for Stirling numbers of the first kind $\begin{bmatrix} n \\ k \end{bmatrix}_F$

$n \setminus k$	1	2	3	4	5	6	7	8	9	10
1	1									
2	1	1								
3	1	2	1							
4	1	3	4	1						
5	1	4	11	7	1					
6	1	5	26	32	12	1				
7	1	6	57	122	92	20	1			
8	1	7	120	423	582	252	33	1		
9	1	8	247	1389	3333	2598	681	54	1	
10	1	9	502	4414	18054	24117	11451	1815	88	1

Table 20. Table for Stirling numbers of the second kind $\begin{Bmatrix} n \\ k \end{Bmatrix}_F$

n\k	1	2	3	4	5	6	7	8	9	10
1	1									
2	4	1								
3	12	4	1							
4	12	16	8	1						
5	48	76	56	14	1					
6	288	504	468	168	24	1				
7	2592	4824	5184	2316	480	40	1			
8	36288	70128	82584	42240	10956	1320	66	1		
9	798336	1579104	1969560	1096344	327096	49236	3564	108	1	
10	27941760	56066976	71062070	45234288	13853088	2395008	216744	9504	176	1

Table 21. Table for $\left]\genfrac{}{}{0pt}{}{n}{k}\right[_F$

$n \setminus k$	1	2	3	4	5	6	7	8	9	Row Sum
1	1									1
2	1	1								2
3	0	2	1							3
4	-2	2	3	1						4
5	-7	5	1	5	1					5
6	-23	24	-8	4	8	1				6
7	-92	145	-80	16	4	13	1			7
8	-518	1101	810	228	-24	9	21	1		1628
9	-4455	2779	-1610	397	253	463	45	12	1	-2115

Table 22. Table for $\binom{n}{k}_F$

It is worth noticing that the row sums equal to the row numbers for the first seven rows of Table 22 and the Fibonacci values for the first nine upward diagonals 1, 1, 1, 0, -4, -15, -66, -376, -3429 are not pretty at all! We can also verify

$$\left]\genfrac{}{}{0pt}{}{n}{k}\right[_F = \left]\genfrac{}{}{0pt}{}{n-1}{k-1}\right[_F + (F_{n-1} + F_k)\left]\genfrac{}{}{0pt}{}{n-1}{k}\right[_F \qquad \text{Eq. 27}$$

with Table 21. In fact, the final thing left to do is to show that this recursive formula holds true for any underlying sequence!

Let $(a_i)_1^\infty$ be any sequence in a commutative ring.

Theorem. If we define $\begin{bmatrix}1\\1\end{bmatrix}_{(a_i)_1^\infty} = \begin{Bmatrix}1\\1\end{Bmatrix}_{(a_i)_1^\infty} = 1$,

$$\begin{bmatrix}n\\k\end{bmatrix}_{(a_i)_1^\infty} = \begin{bmatrix}n-1\\k-1\end{bmatrix}_{(a_i)_1^\infty} + a_{n-1}\begin{bmatrix}n-1\\k\end{bmatrix}_{(a_i)_1^\infty}, \qquad \text{Eq. 28}$$

$$\begin{Bmatrix}n\\k\end{Bmatrix}_{(a_i)_1^\infty} = \begin{Bmatrix}n-1\\k-1\end{Bmatrix}_{(a_i)_1^\infty} + a_k\begin{Bmatrix}n-1\\k\end{Bmatrix}_{(a_i)_1^\infty} \qquad \text{Eq. 29}$$

and

$$\left]\begin{matrix}n\\k\end{matrix}\right[_{(a_i)_1^\infty} = \sum_{j=0}^{n-k}\begin{bmatrix}n\\k+j\end{bmatrix}_{(a_i)_1^\infty}\begin{Bmatrix}k+j\\k\end{Bmatrix}_{(a_i)_1^\infty}, \qquad \text{Eq. 30}$$

then

$$\left]\begin{matrix}n\\k\end{matrix}\right[_{(a_i)_1^\infty} = \left]\begin{matrix}n-1\\k-1\end{matrix}\right[_{(a_i)_1^\infty} + [a_{n-1}+a_k]\left]\begin{matrix}n-1\\k\end{matrix}\right[_{(a_i)_1^\infty}. \qquad \text{Eq. 31}$$

Proof. We shall only prove the case when n=5 by using mathematical induction. For convenience, let a_1= a, a_2= b, a_3= c, a_4= d and a_5= e. We first Eqs. 28 and 29 to tabulate Stirling numbers of both kind as follows.

$n \setminus k$	1	2	3	4	5
1	1				
2	a	1			
3	ab	a+b	1		
4	abc	ab+ac+bc	a+b+c	1	
5	abcd	abc+abd+acd+bcd	ab+ac+ad+bc+bd+cd	a+b+c+d	1

Table 23. Table for $\begin{bmatrix}n\\k\end{bmatrix}_{(a_i)_1^\infty}$

$n \backslash k$	1	2	3	4	5
1	1				
2	a	1			
3	a^2	a+b	1		
4	a^3	a^2+ab+b^2	a+b+c	1	
5	a^4	a^3+a^2b+ab^2+b^3	a^2+ab+ac+b^2+bc+c^2	a+b+c+d	1

Table 24. Table for $\left\{ {n \atop k} \right\}_{(a_i)_1^\infty}$

We then use Tables 23 and 24 via Eq. 30 to tabulate Lah numbers below.

$n \backslash k$	1	2	3	4	5
1	1				
2	2a	1			
3	2a(a+b)	2(a+b)	1		
4	2a(a+b)(a+c)	2(a+b)(a+b+c)	2(a+b+c)	1	
5	2a(a+b)(a+c)(a+d)	(2a+b)(a^2+ab+ac+ad+b^2+bc+bd+cd)	2(a+b+c)(a+b+c+d)	2(a+b+c+d)	1

Table 25. Table for $\left] {n \atop k} \right[_{(a_i)_1^\infty}$

We need only check Eq. 31 for k > 1 and k < n-1. In our case, we need only check

$$\left] {4 \atop 2} \right[_{(a_i)_1^\infty} - \left] {3 \atop 1} \right[_{(a_i)_1^\infty} - (c+b) \left] {3 \atop 2} \right[_{(a_i)_1^\infty} = 2(a+b)(a+b+c) - 2a(a+b) - (c+b)[2(a+b)] = 0 ;$$

$$\left] {5 \atop 2} \right[_{(a_i)_1^\infty} - \left] {4 \atop 1} \right[_{(a_i)_1^\infty} - (c+b) \left] {4 \atop 2} \right[_{(a_i)_1^\infty}$$

$$= (2a+b)(a^2+ab+ac+ad+b^2+bc+bd+cd) - 2a(a+b)(a+c) - (c+b)[2(a+b)(a+b+c)] = 0 ;$$

$$\left] {5 \atop 3} \right[_{(a_i)_1^\infty} - \left] {4 \atop 2} \right[_{(a_i)_1^\infty} - (c+b) \left] {4 \atop 3} \right[_{(a_i)_1^\infty}$$

$$= 2(a+b+c)(a+b+c+d) - 2(a+b)(a+b+c) - (c+b)[2(a+b+c)] = 0 .$$

The inductive step can be carried out by checking the terms involving the new entry e for the case that n=6.

GLOSSARY

Polynomial: A mathematical expression such as ax^3+bx^2-cx, where x is a variable and a, b, c are called coefficients.

Binomial expansion: According to the binomial theorem, it is possible to expand the polynomial $(x + y)^n$ into a sum involving terms of the form ax^by^c, where the exponents b and c are nonnegative integers with $b + c = n$, and the coefficient a of each term is a specific positive integer depending on n and b.

Combinatorics: The branch of mathematics dealing with combinations of objects belonging to a finite set in accordance with certain constraints.

Mathematical induction: To prove a statement S(n) is true for any natural number n, it suffices first to establish the inductive basis [to prove S(1) is true] and then to provide the inductive step [to prove S(m+1) is true by assuming S(m) is true].

Lah numbers: The number of ways to sort the first n terms of the natural sequence into k nonempty linear ordered subsets.

q-Gaussian coefficient: It is also called q-binomial coefficient or q-Gaussian polynomial, which is a q-analog for the binomial coefficient.

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APPENDIX: LIST OF TABLES

Editors of

"EVOLUTIONARY PROGRESS IN SCIENCE, TECHNOLOGY, ENGINEERING, ARTS AND MATHEMATICS (STEAM)"

1. Dr. Lawrence K. Wang (王抗曝)

Editor Lawrence K. Wang has served the society as a professor, inventor, chief engineer, chief editor and public servant (UN, USEPA, NY, Albany) for 30+ years, with experience in entire field of environmental science, technology, engineering arts and mathematics (STEAM). He is a licensed NY-MA-NJ-PA-OH Professional Engineer, a certified NY-MA-RI Laboratory Director, a MA-NY Water Operator, and an OSHA Train-the-Trainer Instructor. He has special passion, and expertise in developing various innovative technologies, educational programs, licensing courses, international projects, academic publications, and humanitarian organizations, all for his dream goal of promoting world peace. He is a retired Acting President/Professor of the Lenox Institute of Water Technology, USA, a United Nations Industrial and Development Organization (UNIDO) Senior Advisor in Austria, and a former professor/visiting professor of Rensselaer Polytechnic Institute, Stevens Institute of Technology, University of Illinois, National CK University, Zhejiang University, and Tongji University. Dr. Wang is the author of 750+ papers and 50+ books, and is credited with 29 invention patents. He holds a BSCE degree from National CK University, a MSCE degree from the University of Missouri, a MS degree from the University of Rhode Island and a PhD degree from Rutgers University, USA. Currently he is the book series editor of CRC Press, Springer Nature Switzerland, Lenox Institute Press, World Scientific Singapore, and John Wiley. Dr. Wang has been a Delegate of the People to People International Foundation, an American Academy of Environmental Engineers (AAEE) Diplomate, a member of ASCE, AIChE, ASPE, WEF, AWWA, CIE and OCEESA, and an awardee of many US and international engineering awards.

2. Dr. Hung-ping Tsao （曹恆平）

Editor Hung-ping Tsao has been a mathematician, a university professor, and an assistant actuary, serving private firms and universities in the United States and Taiwan for 30+ years.d to be an Associate Member of the Society of Actuaries and a Member of the American Mathematical Society. His research have been in the areas of college mathematics, actuarial mathematics, management mathematics, classic number theory and Sudoku puzzle solving. In particular, bikini and open top problems are presented to share some intuitive insights and some type of optimization problems can be solved more efficiently and categorically by using the idea of the boundary being the marginal change of a well-rounded region with respect to its inradius; theory of interest, life contingency functions and pension funding are presented in more simplified and generalized fashions; the new way of the simplex method using cross-multiplication substantially simplified the process of finding the solutions of optimization problems; the generalization of triangular arrays of numbers from the natural sequence based to arithmetically progressive sequences based opens up the dimension of explorations; the introduction of step-by-step attempts to solve Sudoku puzzles makes everybody's life so much easier and other STEAM project development. Dr. Tsao is the author of 3 books and over 30 academic publications. Among all of the above accomplishments, he is most proud of solving manually in the total of ten hours the hardest Sudoku posted online by Arto Inkala in early July of 2012. He earned his high school diploma from the High School of National Taiwan Normal University, his BS and MS degrees from National Taiwan Normal University, Taipei, Taiwan, his second MS degree from the UWM in USA, and a PhD degree from the University of Illinois, USA.

Editors of the eBOOK Series of the *"EVOLUTIONARY PROGRESS INSCIENCE, TECHNOLOGY, ENGINEERING, ARTS AND MATHEMATICS (STEAM)"*

Dr. Lawrence K. Wang (王抗曝) - - left

Dr. Hung-ping Tsao (曹恆平) -- right

E-BOOK SERIES AND CHAPTER INTRODUCTON

Introduction to the E-BOOK Series of the *"EVOLUTIONARY PROGRESS IN SCIENCE, TECHNOLOGY, ENGINEERING, ARTS AND MATHEMATICS (STEAM)"*
and This Chapter "EVOLUTIONARY MATHEMATICS AND SCIENCE FOR ULTIMATE GENERALIZATION OF LAH NUMBERS/(BINOMIAL COEFFICIENTS): SUMS/(ALTERNATE SUMS) OF ORTHOGONAL PRODUCTS OF STIRLING NUMBERS"

The acronym STEM stands for "science, technology, engineering and mathematics". In accordance with the National Science Teachers Association (NSTA), "A common definition of STEM education is an interdisciplinary approach to learning where rigorous academic concepts are coupled with real-world lessons as students apply science, technology, engineering, and mathematics in contexts that make connections between school, community, work, and the global enterprise enabling the development of STEM literacy and with it the ability to compete in the new economy". The problem of this country has been pointed out by the US Department of Education that "All young people should be prepared to think deeply and to think well so that they have the chance to become the innovators, educators, researchers, and leaders who can solve the most pressing challenges facing our nation and our world, both today and tomorrow. But, right now, not enough of our youth have access to quality STEM learning opportunities and too few students see these disciplines as springboards for their careers." STEM learning and applications are very popular topics at present, and STEM related careers are in great demand. According to the US Department of Education reports that the number of STEM jobs in the United States will grow by 14% from 2010 to 2020, which is much faster than the national average of 5-8 % across all job sectors. Computer programming and IT jobs top the list of the hardest to fill jobs.

Despite this, the most popular college majors are business, law, etc., not STEM related. For this reason, the US government has just extended a provision allowing foreign students that are earning degrees in STEM fields a seven month visa extension, now allowing them to stay for up to three years of "on the job training". So, at present STEM is a legal term. The acronym STEAM stands for "science, technology, engineering, arts and mathematics". As one can see, STEAM (adds "arts") is simply a variation of STEM. The word of "arts" means application, creation, ingenuity, and integration, for enhancing STEM inside, or exploring of

STEM outside. It may also mean that the word of "arts" connects all of the humanities through an idea that a person is looking for a solution to a very specific problem which comes out of the original inquiry process. STEAM is an academic term in the field of education.

The University of San Diego and Concordia University offer a college degree with a STEAM focus. Basically STEAM is a framework for teaching or R&D, which is customizable and functional, thence the "fun" in functional. As a typical example, if STEM represents a normal cell phone communication tower looking like a steel truss or concrete column, STEAM will be an artificial green tree with all devices hided, but still with all cell phone communication functions. This e-book series presents the recent evolutionary progress in STEAM with many innovative chapters contributed by academic and professional experts.

This e-book chapter, "EVOLUTIONARY MATHEMATICS AND SCIENCE FOR ULTIMATE GENERALIZATION OF LAH NUMBERS/(BINOMIAL COEFFICIENTS): SUMS/(ALTERNATE SUMS) OF ORTHOGONAL PRODUCTS OF STIRLING NUMBERS" is Dr. Hung-ping Tsao's collection of thoughts, works and articles about various ways of coming up with formulas for sums of powers throughout his retired period for seventeen years now. Three years prior to the publication of "*EXPLICIT POLYNOMIAL EXPRESSIONS FOR SUMS OF POWERS OF AN ARITHMETIC PROGRESSION*", he gave a few talks among universities in Taiwan and a class of gifted students of his Alma Mater (High School of National Taiwan Normal University). He was then invited to present "General Triangular Arrays of Numbers" by "22nd Asian Technology Conference in Mathematics" (Chung Yuan Christian University, December 19, 2017). He is also grateful that Professor Ronald Graham [author of "*CONCRETE MATHEMATICS*"] replied promptly to my e-mails with two separate attachments of his manuscripts that he generalized most of the special functions in Chapter 6 of "*CONCRETE MATHEMATICS*". He is presenting here a systemic but rather long account of his personal excursion into the realm of numbers initiated by Blaise Pascal, James Stirling, Leonhard Euler and Jacob Bernoulli, which is therefore not meant to be a categorical survey of the topic.