1. REPORT DATE (DD-MM-YYYY) 1-31-2021
2. REPORT TYPE NEW RESEARCH REPORT
3. DATES COVERED JAN 2003-JAN 2021
4. TITLE AND SUBTITLE EVOLUTIONARY MATHEMATICS AND SCIENCE FOR GENERAL FAMOUS NUMBERS: STIRLING-EULER-LAH-BELL
5a. CONTRACT NUMBER N/A
5b. GRANT NUMBER N/A
5c. PROGRAM ELEMENT NUMBER N/A
5d. PROJECT NUMBER STEAM-VOL3-NUM1-JAN2021
5e. TASK NUMBER N/A
5f. WORK UNIT NUMBER N/A
6. AUTHOR(S) Tsao, Hung-ping and Chang, Leon
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) TSAO RESIDENCE, 1151 Highland Drive, Novato, CA 94949, USA
8. PERFORMING ORGANIZATION REPORT NUMBER STEAM-VOL3-NUM1-JAN2021
9. SPONSORING / MONITORING AGENCY NAME(S) AND ADDRESS(ES) Wang, Lawrence K. and Tsao, Hung-ping (editors). "Evolutionary Progress in Science, Technology, Engineering, Arts, and Mathematics (STEAM)", Volume 3, Number 1, January 2021; Lenox Institute Press, PO Box 405, Newtownville, NY, 12128-0405, USA
10. SPONSOR/MONITOR’S ACRONYM(S) LENOX
11. SPONSOR/MONITOR’S REPORT NUMBER(S) STEAM-VOL3-NUM1-JAN2021
12. DISTRIBUTION / AVAILABILITY STATEMENT NO RESTRICTION
14. ABSTRACT We first introduce Pascal, Stirling, Eulerian, Lah and Bell numbers via sorting, then generalize Stirling numbers of both kinds \( \begin{pmatrix} n \\ k \end{pmatrix} \) and \( \begin{pmatrix} n \\ k \end{pmatrix} \), Eulerian numbers of two orders \( \begin{pmatrix} n \\ k \end{pmatrix} \) and \( \begin{pmatrix} n \\ k \end{pmatrix} \), Lah numbers \( L(n,k) = \sum_{j=0}^{n} \begin{pmatrix} j \\ k \end{pmatrix} \), and \( \sum_{j=1}^{n} j^2 \begin{pmatrix} n \\ j \end{pmatrix} = \sum_{j=0}^{n} \binom{j+1}{k} \binom{n}{j} \), the right-hand side of which is an ordered Bell polynomial, from the natural sequence based to arithmetically progressive sequences based. We further construct two types of arrays with any infinite sequence based to arithmetically progressive sequences based. We first introduce Pascal, Stirling, Eulerian, Lah and Bell numbers via sorting, then generalize Stirling numbers of both kinds \( \begin{pmatrix} n \\ k \end{pmatrix} \) and \( \begin{pmatrix} n \\ k \end{pmatrix} \), possessing diagonal polynomials. We shall introduce more Eulerian arrays with various initial values \( T(n,0) \) for \( n > 0 \) and prove the existence of the general Stirling polynomials. Python programs are used to produce tables for both types of arrays along with difference tables of their diagonals to facilitate the calculation of diagonal polynomials in the case of existence.
15. SUBJECT TERMS (Keywords) Polynomial expression, Natural sequence, Binomial coefficient, Stirling number, Lah number, Natural sequence, Pascal triangle, Bernoulli coefficient, Arithmetically progressive sequence, Recursive formula, Sorting, Cycle, Subset, Eulerian number, Second generation Stirling numbers, Second generation Eulerian numbers, Stirling polynomial, Diagonal polynomial, q-Gaussian coefficient, Bell number, Ordered Bell polynomial, Python.
16. SECURITY CLASSIFICATION OF: UNCLASSIFIED/UNLIMITED (UU)
a. REPORT UU b. ABSTRACT UU c. THIS PAGE UU
17. LIMITATION OF ABSTRACT UU PAGES 99
18. NUMBER OF ABSTRACT UU PAGES 99
19a. NAME OF RESPONSIBLE PERSON Wang, Lawrence K.
19b. TELEPHONE NUMBER (Include area code) (518) 250-0012

Standard Form 298 (Rev. 8-98)
Prescribed by ANSI St
EVOLUTIONARY MATHEMATICS AND SCIENCE FOR
GENERAL FAMOUS NUMBERS: STIRLING-EULER-LAH-BELL

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We first introduce Pascal, Stirling, Eulerian, Lah and Bell numbers via sorting, then generalize Stirling numbers of both kinds $\left\langle \begin{array}{c} n \\ k \end{array}\right\rangle$, $\left\{\begin{array}{c} n \\ k \end{array}\right\}$, Eulerian numbers of two orders $\langle \left\langle \begin{array}{c} n \\ k \end{array}\right\rangle \rangle$, $\langle \left\langle \begin{array}{c} \langle \begin{array}{c} n \\ k \end{array}\rangle \rangle \end{array}\right\rangle$, Lah numbers $L(n, k) = \sum_{j=1}^{n} \left\{ \begin{array}{c} n \\ j \end{array}\right\} \left\{ \begin{array}{c} j \\ k \end{array}\right\}$ and

$$\sum_{k=0}^{n-1} 2^k \left\langle \begin{array}{c} n \\ k \end{array}\right\rangle = \sum_{k=1}^{n} \left\{ \sum_{j=1}^{k+1} \left(\begin{array}{c} k+1 \\ j \end{array}\right) \right\} \left\{ \begin{array}{c} n \\ k \end{array}\right\},$$

the right-hand side of which is an ordered Bell polynomial, from the natural sequence based to arithmetically progressive sequences based. We further construct two types of arrays with any infinite sequence base $A(i)$: $T(n, k | A(i) = a + (i - 1)d | 1;2)$, $T(n, k | A(i) = a + (i - 1)d | 1;3)$, possessing diagonal (Stirling) polynomials;

$$\langle \left\{\begin{array}{c} n \\ k \end{array}\right\} \rangle = T(n, k + 1 | A(i) = a + (i - 1)d | 5;6),$$

$T(n, 0) = [A(2) - 2A(1)]T(n - 1, 0)$ for $n > 0$.

1. Stirling: $u = 1$, $0 \leq k \leq n$ and the initial values $T(n, 0) = 0$ for $n > 0$.

2. Eulerian: $u > 1$, $-1 \leq k \leq n - 1$ and $T(n, 0)$, $n > 0$, varies for each array.

We shall introduce more Eulerian arrays with various initial values $T(n, 0)$ for $n > 0$ and prove the existence of the general Stirling polynomials. Python programs are used to produce tables for both types of arrays along with difference tables of their diagonals to facilitate the calculation of diagonal polynomials in the case of existence.

**Keywords:** Natural sequence, Binomial coefficient, Stirling number, Lah number, Natural sequence, Pascal triangle, Bernoulli coefficient, Arithmetically progressive sequence, Recursive formula, Sorting, Cycle, Subset, Eulerian number, Second generation Stirling numbers, Second generation Eulerian numbers, Stirling polynomial, Diagonal polynomial, q-Gaussian coefficient, Bell number, Ordered Bell polynomial, Python.
NOMENCLATURE

C(n, k), \( \binom{n}{k} \) combination

\[ \Sigma \] sum

\((i)_n^n\) the natural sequence

P(n, k), the permutation of n elements taken k at a time

k! k factorial

L(n, k) Lah number

\( \begin{pmatrix} n \\ k \end{pmatrix} \) first-order Eulerian number

\( \begin{pmatrix} n \\ k \end{pmatrix} \) second-order Eulerian number

\( \begin{bmatrix} n \\ k \end{bmatrix} \) Stirling number of the first kind

\( \begin{bmatrix} n \\ k \end{bmatrix} \) Stirling number of the second kind

\((a + (i - 1)d)^\infty_i\) arithmetically progressive sequence

\( \begin{bmatrix} n \\ k \end{bmatrix} \) Stirling triangle of the first kind for \((a + (i - 1)d)^\infty_i\)

\( \begin{bmatrix} n \\ k \end{bmatrix} \) Stirling triangle of the second kind for \((a + (i - 1)d)^\infty_i\)

\( \begin{pmatrix} n \\ a,d \end{pmatrix} \) first-order Eulerian number for \((a + (i - 1)d)^\infty_0\)

\( \begin{pmatrix} n \\ a,d \end{pmatrix} \) second-order Eulerian number for \((a + (i - 1)d)^\infty_0\)

\( T^{r,s}_{(a)_{0}^{\infty}} \) general triangular array for \((a_{j})_{0}^{\infty}\)

\[ \prod \] product

\( \begin{pmatrix} a-1 \\ k-q \end{pmatrix} \) \( q \) - Gaussian coefficient

\( B_{n} \) Bell number
1. **INTRODUCTION**

Binomial coefficients $C(n,k)$ can be displayed as Pascal triangle (see Table 1), which was discovered about one thousand years ago by Al-Karaji. In fact, it could trace back to the second century B.C. by Pingala and for the subsequent thousand years there had been documentary evidences that Pascal triangle had been mentioned independently in India, Greece, China and Persia.

\[
\begin{array}{cccccccccccc}
\text{n\backslash k} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
0 & 1 &   &   &   &   &   &   &   &   &   &   \\
1 & 1 & 1 &   &   &   &   &   &   &   &   &   \\
2 & 1 & 2 & 1 &   &   &   &   &   &   &   &   \\
3 & 1 & 3 & 3 & 1 &   &   &   &   &   &   &   \\
4 & 1 & 4 & 6 & 4 & 1 &   &   &   &   &   &   \\
5 & 1 & 5 & 10 & 10 & 5 & 1 &   &   &   &   &   \\
6 & 1 & 6 & 15 & 20 & 15 & 6 & 1 &   &   &   &   \\
7 & 1 & 7 & 21 & 35 & 35 & 21 & 7 & 1 &   &   &   \\
8 & 1 & 8 & 28 & 56 & 70 & 56 & 28 & 8 & 1 &   &   \\
9 & 1 & 9 & 36 & 84 & 126 & 126 & 84 & 36 & 9 & 1 &   \\
10 & 1 & 10 & 45 & 120 & 210 & 252 & 210 & 120 & 45 & 10 & 1 \\
\end{array}
\]

Table 1. Pascal Triangle

As a matter of fact, $C(n,k)$ and $\sum_{i=1}^{n} i^k$ got intertwined again in the Eighteen Century by the famous mathematicians such as Blaise Pascal, James Stirling, Leonhard Euler and Jacob Bernoulli. Our main goal had been to use binomial coefficients $C(k, j)$ in Table 1 to find the general Bernoulli coefficient $b(k, j)$, with $b(k, l)$ denoting Bernoulli numbers, in the following expression
\[
\sum_{j=1}^{n} i^k = \sum_{j=1}^{k-1} b(k, j)n^j,
\]

Eq.1

displayed in the Bernoulli triangle in Table 2.

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<th>3</th>
<th>4</th>
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<td>1/3</td>
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<td>1/4</td>
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<td>1/11</td>
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<td>0</td>
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<td>1/12</td>
<td></td>
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</tr>
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<td>13/12</td>
<td>1/2</td>
<td>1/14</td>
</tr>
</tbody>
</table>

Table 2. Bernoulli triangle
Nothing is more impressive than the Pascal triangle
巴斯卡三角形鮮

It displays those numbers ever so natural and simple
展示數字多自然

I have long dreamed of writing a prospective article
渴望夢想成篇寫

To show the inner beauty of numbers from my angle
數之內斂我見焉

As a matter of fact, the generalization of Eq. 1 had been obtained in (6) as

\[
\sum_{i=1}^{n}[a+(i-1)d]_k = \sum_{j=0}^{k} \sum_{j=k+1-\ell}^{k+1} (-1)^{j-k-1-\ell} \frac{d^{j-1}}{j} \left[ \binom{j}{k+1-\ell} \sum_{a,d} \right] n^{k+1-\ell}, \quad \text{Eq. 2}
\]

which is also denoted as \( S^{(k)}_n \), where each term is a weighted product of a Stirling number of the first kind, with notations adopted from (1), as shown in Table 3 for \( n \) up to 10 and a general Stirling number of the second kind as shown in Table 4 for \( n \) up to 10, \( a = 2, d = 3 \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>( k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
</tr>
<tr>
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</tr>
<tr>
<td>10</td>
<td>362880</td>
</tr>
</tbody>
</table>

Table 3: Table for Stirling numbers of the first kind
To be more specific, we can write

\[
S_n^{(1)} = \frac{d}{2} \binom{2}{2} a_{n,d} n^2 + \binom{1}{1} a_{n,d} - \frac{d}{2} \binom{2}{2} a_{n,d} n, \quad \text{Eq. 3}
\]

\[
S_n^{(2)} = \frac{d^2}{3} \binom{3}{3} a_{n,d} n^3 + \binom{d}{2} \binom{3}{2} a_{n,d} - \frac{d^2}{3} \binom{3}{3} a_{n,d} n^2
\]

\[
+ \binom{d}{2} \binom{3}{2} a_{n,d} \binom{1}{1} a_{n,d} + \frac{d^2}{3} \binom{3}{3} a_{n,d} n, \quad \text{Eq. 4}
\]

\[
S_n^{(3)} = \frac{d^3}{4} \binom{4}{4} a_{n,d} n^4 + \binom{d^2}{2} \binom{4}{2} a_{n,d} - \frac{d^3}{4} \binom{4}{4} a_{n,d} n^3
\]

\[
+ \binom{d^2}{2} \binom{4}{2} a_{n,d} \binom{d}{2} \binom{3}{2} a_{n,d} \binom{1}{1} a_{n,d} + \frac{d^3}{4} \binom{4}{4} a_{n,d} n^2
\]

\[
+ \binom{d}{2} \binom{4}{2} a_{n,d} \binom{1}{1} a_{n,d} - \frac{d^3}{4} \binom{4}{4} a_{n,d} n, \quad \text{Eq. 5}
\]
In the case when $a = d = 1$, Eq. 2 becomes

$$\sum_{i=1}^{n} i^k = \sum_{r=0}^{k} \sum_{j=k+1-r}^{k+1} (-1)^{j-k-1+r} \frac{1}{j!} \sum_{\binom{k+1}{j}} \binom{n}{j} n^{k+1-r}, \quad \text{Eq. 6}$$

where Stirling numbers of the second kind for $n$ up to 10 are shown in Table 5.

<table>
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<th>2</th>
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<th>5</th>
<th>6</th>
<th>7</th>
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<td>22827</td>
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<td>750</td>
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</tbody>
</table>

Table 5. Stirling triangle of the second kind

Amazingly, binomial coefficients $C(n,k)$ can be expressed as

$$C(n,k) = \sum_{j=k}^{n} (-1)^{j-k} \binom{n}{j} \left[ \frac{j}{k} \right] \frac{1}{j!} \sum_{\binom{n+1}{j}} \binom{n}{j} n^{j-k}, \quad \text{Eq. 7}$$

and used in (7) to derive Eq. 6! On the other hand, the Lah number $L(n, k)$, defined in (2) as the number of ways to sort the first $n$ terms of the natural sequence into $k$ nonempty linear ordered subsets, turns out to be

$$L(n, k) = \sum_{j=1}^{n} \binom{n}{j} \left[ \frac{j}{k} \right]. \quad \text{Eq. 8}$$
The number of ways of sorting the first $n$ terms of $(i)_i^n$ into $k$ cycles is the Stirling number of the first kind $\left[ \begin{array}{c} n \\ k \end{array} \right]$. Clearly, $\left[ \begin{array}{c} 1 \\ 1 \end{array} \right] = 1$. There is only one way of sorting the first 2 terms of $(i)_i^n$ into 1 cycle, namely [1,2], since there is no distinction made between [1,2] and [2,1]. Hence $\left[ \begin{array}{c} 2 \\ 1 \end{array} \right] = 1$. Also, there is only one way of sorting the first 2 terms of $(i)_i^n$ into 2 cycles, namely [1] and [2], so that $\left[ \begin{array}{c} 2 \\ 2 \end{array} \right] = 1$. There are two ways of sorting the first 3 terms of $(i)_i^n$ into 1 cycle, namely [1,2,3] and [1,3,2] so that $\left[ \begin{array}{c} 3 \\ 1 \end{array} \right] = 2$. There are three ways of sorting the first 3 terms of $(i)_i^n$ into 2 cycles, namely [1,2], [1,3] and [2,3] so that $\left[ \begin{array}{c} 3 \\ 2 \end{array} \right] = 3$. There is only one way of sorting the first 3 terms of $(i)_i^n$ into 3 cycles, namely [1], [2] and [3] so that $\left[ \begin{array}{c} 3 \\ 3 \end{array} \right] = 1$.

In general, sorting the first $n$ terms of $(i)_i^n$ into $k$ cycles, there are $\left[ \begin{array}{c} n-1 \\ k-1 \end{array} \right]$ ways including [n], since the number of ways of sorting the first $n-1$ terms into $k-1$ cycles is $\left[ \begin{array}{c} n-1 \\ k-1 \end{array} \right]$. For any one of the $\left[ \begin{array}{c} n-1 \\ k \end{array} \right]$ ways of sorting, not including $n$, we need to insert $n$ into 1 (say, a $j_i$-cycle) of the $k$ cycles. Since there are $j_i$ ways of doing such insertion, the total possible ways of inserting $n$ into any of those $k$ cycles is $\sum_{i=1}^{k} j_i = n - 1$. Thus we have

$$\left[ \begin{array}{c} n \\ k \end{array} \right] = \left[ \begin{array}{c} n-1 \\ k-1 \end{array} \right] + (n-1) \left[ \begin{array}{c} n-1 \\ k \end{array} \right].$$

Eq. 9
The number of ways of sorting the first \( n \) terms of \( (i)_k^\infty \) into \( k \) sets is the Stirling number of the second kind \( \{ n \}_{k} \). Clearly, \( \{ 1 \}_{1} = 1 \). There is only one way of sorting the first 2 terms of \( (i)_1^\infty \) into 1 set, namely \( \{ 1,2 \} \), so that \( \{ 2 \}_{1} = 1 \). Also, there is only one way of sorting the first 2 terms of \( (i)_2^\infty \) into 2 sets, namely \( \{ 1 \} \) and \( \{ 2 \} \), so that \( \{ 2 \}_{2} = 1 \).

There is only one way of sorting the first 3 terms of \( (i)_1^\infty \) into 1 set, namely \( \{ 1,2,3 \} \), so that \( \{ 3 \}_{1} = 1 \). There are three ways of sorting the first 3 terms of \( (i)_2^\infty \) into 2 sets, namely \( \{ 1,2 \} \), \( \{ 3 \} \); \( \{ 1,3 \}, \{ 2 \} \); and \( \{ 2,3 \}, \{ 1 \} \) so that \( \{ 3 \}_{2} = 3 \). There is only one way of sorting the first 3 terms of \( (i)_3^\infty \) into 3 sets, namely \( \{ 1 \} \), \( \{ 2 \} \) and \( \{ 3 \} \) so that \( \{ 3 \}_{3} = 1 \).

In general, sorting the first \( n \) terms of \( (i)_k^\infty \) into \( k \) sets, there are two cases to consider.

**Case 1.** There are \( \{ n-1 \}_{k-1} \) ways if the singleton \( \{ n \} \) is included in the sorted arrangements, since the number of ways of sorting the first \( n-1 \) terms into \( k-1 \) sets is \( \{ n-1 \}_{k-1} \).

**Case 2.** There are \( k \{ n-1 \}_{k} \) ways if the singleton \( \{ n \} \) is not included in the sorted arrangements, since for any one of \( \{ n-1 \}_{k} \) ways of sorting the term \( n \) can be inserted into any one of those \( k \) sets.

Thus we have proved the recursive formula

\[
\{ n \}_{k} = \{ n-1 \}_{k-1} + k \{ n-1 \}_{k},
\]

Eq. 10
The first-order Eulerian number $\left\langle \begin{array}{c} a \\ k \end{array} \right\rangle$ is the number of permutations $p_1p_2...p_n$ of the set $\{1,2,...n\}$ that have $k$ ascents, i.e. $k$ places where $p_j < p_{j+1}$. Let us first look at simple examples: 1 gives $\left\langle \begin{array}{c} 1 \\ 0 \end{array} \right\rangle = 1$ and $\left\langle \begin{array}{c} 1 \\ 1 \end{array} \right\rangle = 0$; 21 gives $\left\langle \begin{array}{c} 2 \\ 0 \end{array} \right\rangle = 1$; 12 gives $\left\langle \begin{array}{c} 2 \\ 1 \end{array} \right\rangle = 1$ and $\left\langle \begin{array}{c} 2 \\ 2 \end{array} \right\rangle = 0$; 321 gives $\left\langle \begin{array}{c} 3 \\ 0 \end{array} \right\rangle = 1$; 132, 213, 231, 312 gives $\left\langle \begin{array}{c} 3 \\ 1 \end{array} \right\rangle = 4$; 123 gives $\left\langle \begin{array}{c} 3 \\ 2 \end{array} \right\rangle = 1$ and $\left\langle \begin{array}{c} 3 \\ 3 \end{array} \right\rangle = 0$; 4321 gives $\left\langle \begin{array}{c} 4 \\ 0 \end{array} \right\rangle = 1$; 1234 gives $\left\langle \begin{array}{c} 4 \\ 1 \end{array} \right\rangle = 1$ and $\left\langle \begin{array}{c} 4 \\ 2 \end{array} \right\rangle = 0$; 1432, 2143, 2431, 3142, 3214, 3412, 3241, 4132, 4213, 4231, 4312, 1243, 1324, 1342, 1423, 2134, 2314, 2341, 2413, 3124, 3412, 4123, 4341 give $\left\langle \begin{array}{c} 4 \\ 3 \end{array} \right\rangle = 11$ and $\left\langle \begin{array}{c} 4 \\ 4 \end{array} \right\rangle = 0$.

In general, for a permutation $p_1p_2...p_n$ of $\{1,2,...n\}$ with $k - 1$ ascents, we have two cases to consider.

**Case 1.** We can insert $n$ into $p_1p_2...p_{n-1}$ either after $p_{n-1}$ or between $p_{j-1}$ and $p_j$ whenever $P_{j-1} > P_j$ to form a permutation of $\{1,2,...n\}$ that increases the number of ascents by 1 so that the total number of permutations of $\{1,2,...n\}$ that have $k$ ascents in this case is $(n-k)\left\langle \begin{array}{c} n-1 \\ k-1 \end{array} \right\rangle$. 

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Case 2. For a permutation $p_1p_2\ldots p_{n-1}$ of $\{1,2,\ldots,n\}$ with $k$ ascents, we can insert $n$ into $p_1p_2\ldots p_{n-1}$ either before $p_1$ or between $p_{j-1}$ and $p_j$ whenever $p_{j-1} < p_j$ to form a permutation of $\{1,2,\ldots,n\}$ that maintains the same number of ascents so that the total number of permutations of $\{1,2,\ldots,n\}$ that have $k$ ascents in this case is $(k+1)^{\binom{n-1}{k}}$.

Therefore, we obtain

$$\langle\binom{n}{k}\rangle = (n-k)^{\binom{n-1}{k}} + (k+1)^{\binom{n-1}{k}},$$

Eq. 11

The second-order Eulerian number $\langle\binom{n}{k}\rangle$ is the number of permutations $p_1p_2\ldots p_n$ of the multiset $\{1,1,2,\ldots,n,n\}$ that have $k$ ascents, i.e. $k$ places where $p_j < p_{j+1}$, provided that all numbers between the two occurrences of $m$ are greater than $m$ for $1 \leq m \leq n$.

Here are some simple cases: 11 gives $\langle\binom{1}{0}\rangle = 1$ and $\langle\binom{1}{1}\rangle = 0$; 221 gives $\langle\binom{2}{0}\rangle = 1$;

1122,12211 gives $\langle\binom{2}{0}\rangle = 2$; 332211 gives $\langle\binom{3}{0}\rangle = 1$; 112233 gives $\langle\binom{3}{3}\rangle = 0$;

113322,133221,221133,221331,223311,233211,331122,331221 gives $\langle\binom{3}{1}\rangle = 8$ and

112233,112332,122133,122331,123321,133122 gives $\langle\binom{3}{2}\rangle = 6$. 15
In general, for a permutation $p_1p_2\ldots p_{2n-2}$ of $\{1,1,2,\ldots, n-1, n-1\}$ with $k-1$ ascents, we can insert $n,n$ into $p_1p_2\ldots p_{2n-2}$ either after $p_{n-1}$ or between $p_{j-1}$ and $p_j$ whenever $p_{j-1} \geq p_j$ to form a permutation of $\{1,1,2,\ldots,n,n\}$ that increases the number of ascents by 1 so that the total number of permutations of $\{1,1,2,\ldots,n,n\}$ that have $k$ ascents in this case is $(2n-1-k)\binom{n-1}{k}$; whereas for a permutation $p_1p_2\ldots p_{2n-2}$ of $\{1,1,2,\ldots,n-1,n-1\}$ with $k$ ascents, we can insert $n,n$ into $p_1p_2\ldots p_{2n-2}$ either before $p_{1}$ or between $p_{j-1}$ and $p_j$ whenever $p_{j-1} < p_j$ to form a permutation of $\{1,1,2,\ldots,n,n\}$ that maintains the same number of ascents so that the total number of permutations of $\{1,1,2,\ldots,n,n\}$ that have $k$ ascents in this case is $(k+1)\binom{n-1}{k}$.

Therefore, we obtain

$$\binom{n}{k} = (2n-1-k)\binom{n-1}{k-1} + (k+1)\binom{n-1}{k},$$  \hspace{1cm} \text{Eq. 12}$$

The Bell number $B_n$, defined as the number of ways to sort the first $n$ terms of the natural sequence into different partitions, turns out to be $B_n = \sum_{k=1}^{n} \binom{n}{k}$. Furthermore, the ordered Bell number $OB_n$, defined as $OB_n = \sum_{k=1}^{n} k! \binom{n}{k}$, turns out to be equal to $\sum_{k=0}^{n-1} 2^k \binom{n}{k}$. 

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We shall generalize 
, 
, 
 and 
 for \((a + (i - 1)d)\).

For \((a + (i - 1)d)\), the Stirling triangle of the first kind \(\begin{bmatrix} n \\ k \end{bmatrix}_{a,d}\) can be constructed via

\[
\begin{bmatrix} n \\ k \end{bmatrix}_{a,d} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{a,d} + [a + (n - 2)d]\begin{bmatrix} n-1 \\ k \end{bmatrix}_{a,d}
\]

Eq. 13

with \(\begin{bmatrix} 1 \\ 1 \end{bmatrix}_{a,d} = 1\) as in Table 6, which agrees with Table 3 when \(a = 2\) and \(d = 3\).

<table>
<thead>
<tr>
<th>(n/k)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>(a)</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>(a(a + d))</td>
<td>(2a + d)</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>(a(a + d)(a + 2d))</td>
<td>(3a^2 + 6ad + 2d^2)</td>
<td>(3a + 3d)</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>(a(a + d)(a + 2d)(a + 3d))</td>
<td>(4a^3 + 18a^2d + 22ad^2 + 6d^3)</td>
<td>(6a^2 + 18ad + 11d^2)</td>
<td>(4a + 6d)</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 6. Table for general Stirling numbers of the first kind \(\begin{bmatrix} n \\ k \end{bmatrix}_{a,d}\)

The Stirling triangle of the second kind \(\begin{bmatrix} n \\ k \end{bmatrix}_{a,d}\) can be constructed via

\[
\begin{bmatrix} n \\ k \end{bmatrix}_{a,d} = \begin{bmatrix} n-l \\ k-l \end{bmatrix}_{a,d} + [a + (k - 1)d]\begin{bmatrix} n-l \\ k \end{bmatrix}_{a,d}
\]

Eq. 14

shown in Table 7 which agrees with Table 4 when \(a = 2\) and \(d = 3\).
Lastly, we shall generalize the first-order Eulerian numbers \( \left\langle \frac{n}{k} \right\rangle \) for \( (a + (n-1)d)_{n=1}^\infty \). From Eqs. 3-5, we can come up with

\[
\sum_{i=1}^{n} (a + (i-1)d) = (d - a)\left(\begin{array}{c} n \\ 2 \end{array}\right) + a\left(\begin{array}{c} n+1 \\ 2 \end{array}\right), \tag{15}
\]

\[
\sum_{i=1}^{n} (a + (i-1)d)^2 = (d - a)^2\left(\begin{array}{c} n \\ 3 \end{array}\right) + (-2a^2 + 2ad + d^2)\left(\begin{array}{c} n+1 \\ 3 \end{array}\right) + a^2\left(\begin{array}{c} n+2 \\ 3 \end{array}\right), \tag{16}
\]

and

\[
\sum_{i=1}^{n} (a + (i-1)d)^3 = (d - a)^3\left(\begin{array}{c} n \\ 4 \end{array}\right) + (3a^3 - 6a^2d + 4d^3)\left(\begin{array}{c} n+1 \\ 4 \end{array}\right) \\
+ (-3a^3 + 3a^2d + 3ad^2 + d^3)\left(\begin{array}{c} n+2 \\ 4 \end{array}\right) + a^3\left(\begin{array}{c} n+3 \\ 4 \end{array}\right). \tag{17}
\]

Now, we can define \( \left\langle \frac{n}{k} \right\rangle_{a,d} \) according to Eqs. 15-17.
By virtue of Eq. 15, we define \( \langle 1 \rangle_{a,d}^{1} = d - a \) and \( \langle 1 \rangle_{a,d}^{0} = a \) so that \( \langle 1 \rangle_{a,d}^{1} = 0 \).

Unlike \( \langle n \rangle_{k} \), we start with \( k = -1 \) for \( \langle n \rangle_{a,d}^{k} \). Due to Eqs. 16 and 17, we can define

\[
\begin{align*}
\langle -1 \rangle_{a,d}^{2} &= (d - a)^2, \\
\langle 0 \rangle_{a,d}^{2} &= -2a^2 + 2ad + d^2, \\
\langle 1 \rangle_{a,d}^{2} &= a^2, \\
\langle -1 \rangle_{a,d}^{3} &= (d - a)^3, \\
\langle 0 \rangle_{a,d}^{3} &= 3a^3 - 6a^2d + 4d^3, \\
\langle 1 \rangle_{a,d}^{3} &= -3a^3 + 3a^2d + 3ad^2 + d^3 \quad \text{and} \quad \langle 2 \rangle_{a,d}^{3} = a^3.
\end{align*}
\]

Accordingly, we can generalize Eq. 11 as follows. For our purpose, let us first define

\[
\langle 0 \rangle_{a,d}^{1} = 1. \quad \text{Then we write} \quad \langle 2 \rangle_{a,d}^{1} = (-a + d)\langle 1 \rangle_{a,d}^{1},
\]

\[
\begin{align*}
\langle 0 \rangle_{a,d}^{2} &= (a + d)\langle 1 \rangle_{a,d}^{1} + (-a + 2d)\langle 1 \rangle_{a,d}^{0}, \\
\langle 1 \rangle_{a,d}^{2} &= a\langle 0 \rangle_{a,d}^{1}, \\
\langle 2 \rangle_{a,d}^{1} &= (-a + d)\langle 2 \rangle_{a,d}^{1}, \\
\langle 0 \rangle_{a,d}^{3} &= (a + 2d)\langle 2 \rangle_{a,d}^{1} + (-a + 2d)\langle 2 \rangle_{a,d}^{0}, \\
\langle 1 \rangle_{a,d}^{3} &= (a + d)\langle 2 \rangle_{a,d}^{1} + (-a + 3d)\langle 2 \rangle_{a,d}^{1} \quad \text{and} \\
\langle 2 \rangle_{a,d}^{3} &= a\langle 2 \rangle_{a,d}^{1}.
\end{align*}
\]

In general, we have

\[
\sum_{i=1}^{n} [a + (i-1)d]^{k} = \sum_{j=1}^{k+1} \binom{k}{j} \binom{n+j-1}{k+1}.
\]

As we can check, the above are the special cases of

\[
\langle n \rangle_{a,d}^{k} = [a + (n-k-1)d]\langle k-1 \rangle_{a,d}^{n-1} + [-a + (k+2)d]\langle k-1 \rangle_{a,d}^{n-1}, \quad \text{Eq. 18}
\]

which is the generalization of Eq. 11 and can be used to tabulate \( \langle n \rangle_{a,d}^{k} \) in Table 8.
Table 8. Table for general first order Eulerian numbers $\left\langle \begin{array}{c} n \\ k \end{array} \right\rangle_{a,d}$

Likewise, by defining $\left\langle \begin{array}{c} 0 \\ -1 \end{array} \right\rangle_{a,d} = 1$, $\left\langle \begin{array}{c} 1 \\ -1 \end{array} \right\rangle_{a,d} = d - a$ and $\left\langle \begin{array}{c} 1 \\ 0 \end{array} \right\rangle_{a,d} = a$, we can use

$$\left\langle \begin{array}{c} n \\ k \end{array} \right\rangle_{a,d} = [a + (2n - 2 - k)d] \left\langle \begin{array}{c} n-1 \\ k-1 \end{array} \right\rangle_{a,d} + [-a + (k + 2)d] \left\langle \begin{array}{c} n-1 \\ k \end{array} \right\rangle_{a,d}$$

Eq. 19
to tabulate $\left\langle \begin{array}{c} n \\ k \end{array} \right\rangle_{a,d}$ in Table 9.

Table 9. Table for general second order Eulerian numbers $\left\langle \begin{array}{c} n \\ k \end{array} \right\rangle_{a,d}$
4. PROPAGATION

We can write

\[
\begin{bmatrix}
\binom{n}{n-2}
\end{bmatrix} = 3\binom{n+1}{n-3} - \binom{n}{n-3},
\begin{bmatrix}
\binom{n}{n-3}
\end{bmatrix} = 15\binom{n+2}{n-4} - 10\binom{n+1}{n-4} + \binom{n}{n-4},
\begin{bmatrix}
\binom{n}{n-4}
\end{bmatrix} = 105\binom{n+3}{n-5} - 105\binom{n+2}{n-5} + 25\binom{n+1}{n-5} - \binom{n}{n-5},
\]

which can be proved by mathematical induction in virtue of Eq. 9 as follows.

\[
\begin{bmatrix}
\binom{n}{n-4}
\end{bmatrix} = (n-1)\begin{bmatrix}
\binom{n-1}{n-4}
\end{bmatrix} + \begin{bmatrix}
\binom{n-1}{n-5}
\end{bmatrix}
\]

\[
= (n-1)\left\{15\binom{n}{n-5} - 10\binom{n-1}{n-5}\right\} + 105\binom{n+2}{n-6} - 105\binom{n+1}{n-6} + 25\binom{n}{n-6} - \binom{n-1}{n-6}
\]

\[
= \left\{105\binom{n+2}{n-5} - 45\binom{n+1}{n-5}\right\} + \left\{60\binom{n+1}{n-5} - 20\binom{n}{n-5}\right\} + \left\{5\binom{n}{n-5} - \binom{n-1}{n-5}\right\}
\]

\[
+ 105\binom{n+2}{n-6} - 105\binom{n+1}{n-6} + 25\binom{n}{n-6} - \binom{n-1}{n-6}
\]

\[
= 105\binom{n+3}{n-5} - 105\binom{n+2}{n-5} + 25\binom{n+1}{n-5} - \binom{n}{n-5}.
\]

By closely studying the above inductive step and writing

\[
\begin{bmatrix}
\binom{n}{n-k}
\end{bmatrix} = \sum_{j=1}^{k} S_1(k, j)\binom{n+j-1}{n-k-1},
\]

Eq. 20

with \( S_1(k,1) = (-1)^{k-1} \) and \( S_1(k,k) = \prod_{i=1}^{k} (2i - 1) \), we obtain

\[
\begin{bmatrix}
\binom{n}{n-k}
\end{bmatrix} = \sum_{j=1}^{k} [(k + j - 1)S_1(k-1, j - 1) - jS_1(k-1, j)]\binom{n+j-1}{n-k-1}.
\]
Accordingly, we can arrive at

\[
\begin{bmatrix}
\binom{n}{n-5}
\end{bmatrix} = \binom{n}{n} - 56\binom{n+1}{n-6} + 490\binom{n+2}{n-6} - 1260\binom{n+3}{n-6} + 945\binom{n+4}{n-6},
\]

\[
\begin{bmatrix}
\binom{n}{n-6}
\end{bmatrix} = -\binom{n}{n-7} + 119\binom{n+1}{n-7} - 1918\binom{n+2}{n-7} + 9450\binom{n+3}{n-7} - 17325\binom{n+4}{n-7} + 1039\binom{n+5}{n-7}, \ldots
\]

In fact, we have obtained a second generation Stirling number \( \left[ \begin{bmatrix} k \\ j \end{bmatrix} \right] = S_1(k, j) \), which we shall call the second-order Stirling number of the first kind and can be tabulated via

\[
\left[ \begin{bmatrix} k \\ j \end{bmatrix} \right] = (k + j - 1)\left[ \begin{bmatrix} k-1 \\ j-1 \end{bmatrix} \right] - j\left[ \begin{bmatrix} k-1 \\ j \end{bmatrix} \right].
\]

Eq. 21

Proceeding as before, we have

\[
\left\{ \begin{bmatrix} n \\ n-3 \end{bmatrix} \right\} = \binom{n}{n-4} + 10\binom{n}{n-5} + 15\binom{n}{n-6},
\]

\[
\left\{ \begin{bmatrix} n \\ n-4 \end{bmatrix} \right\} = \binom{n}{n-5} + 25\binom{n}{n-6} + 105\binom{n}{n-7} + 105\binom{n}{n-8},
\]

\[
\left\{ \begin{bmatrix} n \\ n-5 \end{bmatrix} \right\} = \binom{n}{n-6} + 56\binom{n}{n-7} + 490\binom{n}{n-8} + 1260\binom{n}{n-9} + 945\binom{n}{n-10},
\]

\[
\left\{ \begin{bmatrix} n \\ n-6 \end{bmatrix} \right\} = \binom{n}{n-7} + 119\binom{n}{n-8} + 1918\binom{n}{n-9} + 9450\binom{n}{n-10} + 17325\binom{n}{n-11} + 1039\binom{n}{n-12}, \ldots
\]

and eventually arrive at

\[
\left\{ \begin{bmatrix} n \\ n-k \end{bmatrix} \right\} = \sum_{j=1}^{k} S_2(k, j)\binom{n}{n-k-j}, \quad \text{Eq. 22}
\]
where the second generation number \( \binom{k}{j} \) will be called the second-order Stirling number of the second kind.

Quite amazingly and amusingly \( \binom{k}{j} = (-1)^{k-j} \binom{k}{j} \), but which is not true in general as we shall see next.

Since \( \binom{n}{a,d}_{n-k,a,d} = \binom{n-1}{1,a} + \binom{n-1}{2,a} d = (a-d) \binom{n-1}{1,a} + d \binom{n}{2,a} \), we assume the extension of Eq. 20 as

\[
\binom{n}{n-k,a,d} = \sum_{j=1}^{k-1} S_{a,d}^{n-j} (k, j) \binom{n+j}{k+j+1}
\]

Eq. 23

so that \( S_{a,d}^{n-1}(1,-1) = a - d \), \( S_{a,d}^{n-1}(1,0) = d \). Taking \( k = 2 \) and \( n = 3, 4, 5 \) in Eq. 23, we have

\[
S_{a,d}^{n} (2,1) \binom{2}{2} + S_{a,d}^{n} (2,0) \binom{3}{3} + S_{a,d}^{n} (2,1) \binom{4}{4} = \binom{3}{1,a,d} = a(a + d).
\]

\[
S_{a,d}^{n} (2,1) \binom{3}{2} + S_{a,d}^{n} (2,0) \binom{4}{3} + S_{a,d}^{n} (2,1) \binom{5}{4} = \binom{4}{2,a,d} = 3a^2 + 6ad + 2d^2.
\]

\[
S_{a,d}^{n} (2,1) \binom{4}{2} + S_{a,d}^{n} (2,0) \binom{5}{3} + S_{a,d}^{n} (2,1) \binom{6}{4} = \binom{5}{2,a,d} = 6a^2 + 18ad + 11d^2.
\]

from which we obtain

\[
S_{a,d}^{n} (2,1) = (a - d)^2, \quad S_{a,d}^{n} (2,0) = d(3a - 4d) \quad \text{and} \quad S_{a,d}^{n} (2,1) = 3d^2.
\]
A better way is to use Eq. 13 by writing

\[
\begin{align*}
\left[ n+1 \right]_{n-2,a,d} &= [a + (n-1)d] \left[ n \right]_{n-2,a,d} + \left[ n \right]_{n-3,a,d} \\
&= [(a - d) + nd] \left( n-1 \right) \\
&+ [(a - 2d) + (n + 1)d] \left[ d(3a - 4d) \left( n \right)_{n-3} \right] \\
&+ [(a - 3d) + (n + 2)d] \left[ 3d^2 \left( n \right)_{n-3} \right] \\
&+ S_{i}^{a,d}(4,1) \left( n-1 \right)_{n-4} + S_{i}^{a,d}(4,2) \left( n \right)_{n-4} + S_{i}^{a,d}(4,3) \left( n+1 \right)_{n-4} + S_{i}^{a,d}(4,4) \left( n+2 \right)_{n-4} \\
&= (a - d)^3 \left( n-1 \right)_{n-3} + \left[ d(6a^2 - 16ad + 11d^2) \left( n \right)_{n-3} + S_{i}^{a,d}(4,2) \left( n \right)_{n-4} \right] \\
&+ \left[ d(3a^2 + ad + 13d^2) \left( n+1 \right)_{n-3} + S_{i}^{a,d}(4,3) \left( n+1 \right)_{n-4} \right] \\
&+ \left[ 15d^3 \left( n+1 \right)_{n-3} + S_{i}^{a,d}(4,4) \left( n+1 \right)_{n-4} \right].
\end{align*}
\]

from which we obtain \( S_{i}^{a,d}(3,-1) = (a - d)^3 \), \( S_{i}^{a,d}(3,0) = d(6a^2 - 16ad + 11d^2) \),

\( S_{i}^{a,d}(3,1) = 5d^2(3a - 5d) \) and \( S_{i}^{a,d}(3,2) = 15d^3 \) due to Eq. 20.

In addition to

\[
S_{i}^{a,d}(k,1) = (a - d)S_{i}^{a,d}(k-1,1) \quad \text{and} \quad S_{i}^{a,d}(k,k) = (2k-1)dS_{i}^{a,d}(k-1,k-1),
\]
the procedure displayed above is exactly the inductive step of mathematical induction for proving

\[
\left[ \begin{array}{c} n \\ n-3 \end{array} \right]_{a;d} = (a - d)^3 \binom{n}{n-3} + d(6a^2 - 16ad + 11d^2) \binom{n+1}{n-3} \\
+ d(3a^2 + ad + 13d^2) \binom{n+2}{n-3} + 15d^3 \binom{n+3}{n-3}
\]

and can also be used to derive the general recursive formula

\[
S_1^{a;d}(k, j) = (k + j)dS_1^{a;d}(k - 1, j) + [a - (j + 2)d]S_1^{a;d}(k - 1, j - 1) \quad \text{Eq. 24}
\]

for \( k > j > 1 \).

Accordingly, we obtain

\[
\left[ \begin{array}{c} n+1 \\ n-4 \end{array} \right]_{a;d} = (a - d)^4 \binom{n}{n-4} + d(10a^3 - 40a^2 d + 55ad^2 - 26d^3) \binom{n+1}{n-4} \\
+ 5d^2 (9a^2 - 30ad + 26d^2) \binom{n+2}{n-4} + 105d^3 (a - 2d) \binom{n+3}{n-4} + 105d^4 \binom{n+4}{n-4}.
\]

In fact, we have obtained a second generation Stirling number \( \left[ \begin{array}{c} k \\ j \end{array} \right]_{a;d} = S_1^{a;d}(k, j) \) for \( (a + (n - 1)d)^\infty \), which we shall call the second-order Stirling number of the first kind and

the generalization of Eq. 21 is

\[
\left[ \begin{array}{c} k \\ j \end{array} \right]_{a;d} = (k + j)d \left[ \begin{array}{c} k-1 \\ j-1 \end{array} \right]_{a;d} + [a - (j + 2)d] \left[ \begin{array}{c} k-1 \\ j \end{array} \right]_{a;d} \quad \text{Eq. 25}
\]
Next, we shall come up with the second-order Stirling numbers of the second kind in the same manner. Similar to Eq. 23, we can find the extension of Eq. 22 to be

\[
\left\{ \begin{array}{l} \frac{n}{n-k} \\
\end{array} \right\}_{a,d} = \sum_{j=1}^{k-1} S_2^{a,d}(k, j) \left\{ \begin{array}{l} n+j \\
\end{array} \right\}_{n-k}.
\]

Eq. 26

Accordingly, we have \( S_2^{a,d}(1,-1) = a - d \), \( S_2^{a,d}(1,0) = d \), \( S_2^{a,d}(2,-1) = (a - d)(a - 2d) \),

\( S_2^{a,d}(2,0) = d(3a - 2d) \), \( S_2^{a,d}(2,1) = 3d^2 \), \( S_2^{a,d}(3,-1) = (a - d)(a - 2d)(a - 3d) \),

\( S_2^{a,d}(3,0) = d(6a^2 - 11ad + 6d^2) \), \( S_2^{a,d}(3,1) = 5d^2(3a - d) \), \( S_2^{a,d}(3,2) = 15d^3 \)

and in general

\[
S_2^{a,d}(k, j) = (k + j - 3)dS_2^{a,d}(k - 1, j - 1) + [a + (j - 2)d]S_2^{a,d}(k - 1, j),
\]

Eq. 27

with \( S_2^{a,d}(k, k) = 1 \) and \( S_2^{a,d}(k, j) = 0 \) for \( j < -1 \) and \( j > k \). Thus, we have obtained a second generation Stirling number \( \left\{ \begin{array}{l} \frac{k}{j} \\
\end{array} \right\}_{a,d} = S_2^{a,d}(k, j) \), which we shall call the second-order Stirling number of the second kind with the recursion

\[
\left\{ \begin{array}{l} \frac{k}{j} \\
\end{array} \right\}_{a,d} = (k + j - 3)d\left\{ \begin{array}{l} \frac{k-1}{j-1} \\
\end{array} \right\}_{a,d} + [a + (j - 2)d]\left\{ \begin{array}{l} \frac{k-1}{j} \\
\end{array} \right\}_{a,d}.
\]

Eq. 28

Just as the first-order Eulerian number \( \left\{ \begin{array}{l} \frac{n}{k} \\
\end{array} \right\}_{a,d} \) was defined based on

\[
\sum_{i=1}^{n} [a + (i - 1)d]^i = \sum_{j=1}^{k-1} \left\{ \begin{array}{l} \frac{k}{j} \\
\end{array} \right\}_{a,d} \left( \frac{a + j + 1}{k + 1} \right),
\]

Eq. 29

we first define \( \left\{ \begin{array}{l} \frac{0}{-1} \\
\end{array} \right\}_{a,d} = 1 \), \( \left\{ \begin{array}{l} \frac{1}{-1} \\
\end{array} \right\}_{a,d} = d \), \( \left\{ \begin{array}{l} \frac{1}{0} \\
\end{array} \right\}_{a,d} = a \), \( \left\{ \begin{array}{l} \frac{2}{-1} \\
\end{array} \right\}_{a,d} = 2d^2 \), \( \left\{ \begin{array}{l} \frac{2}{0} \\
\end{array} \right\}_{a,d} = (a + d)^2 - a^2 \),

\( \left\{ \begin{array}{l} \frac{3}{-1} \\
\end{array} \right\}_{a,d} = a^2 \), \( \left\{ \begin{array}{l} \frac{3}{0} \\
\end{array} \right\}_{a,d} = 6d^3 \), \( \left\{ \begin{array}{l} \frac{3}{1} \\
\end{array} \right\}_{a,d} = 6d^2(a + d) \), \( \left\{ \begin{array}{l} \frac{3}{2} \\
\end{array} \right\}_{a,d} = (a + d)^3 - a^3 \), \( \left\{ \begin{array}{l} \frac{3}{3} \\
\end{array} \right\}_{a,d} = a^3 \) from

26
\[
\sum_{i=1}^{n}[a+(i-1)d] = d\binom{n}{2} + a\binom{n}{1},
\]
\[
\sum_{i=1}^{n}[a+(i-1)d]^2 = 2d^2\binom{n}{3} + [(a+d)^2 - a^2]\binom{n}{2} + a^2\binom{n}{1},
\]
\[
\sum_{i=1}^{n}[a+(i-1)d]^3 = 6d^3\binom{n}{4} + 6d^2(a+d)\binom{n}{3} + [(a+d)^3 - a^3]\binom{n}{2} + a^3\binom{n}{1}
\]
and in general
\[
\sum_{i=1}^{n}[a+(i-1)d]^k = \sum_{j=1}^{k-1} \binom{k}{j} a^j d^{k-j},
\]
which is comparable to Eq. 29.

To find the recursive formula for \( \binom{n}{k}^{*} a_{ad} \), we write

\[
\binom{2}{-1}^{*} = 2d\binom{1}{-1}^{*}, \quad \binom{2}{0}^{*} = 2d\binom{1}{0}^{*} + a\binom{1}{-1}^{*},
\]
\[
\binom{3}{1}^{*} = 3d\binom{2}{1}^{*} + (a+2d)\binom{2}{0}^{*}, \quad \binom{3}{2}^{*} = d\binom{2}{1}^{*} + a\binom{2}{0}^{*},
\]

from which we see for \( n \geq 2 \) that
\[
\binom{n}{k}^{*} a_{ad} = [a + (n-1-k)d]\binom{n-1}{k-1}^{*} a_{ad} + (n-1-k)d\binom{n-1}{k}^{*} a_{ad}.
\]

Note that
\[
\binom{n}{k}^{*} a_{ad} = \sum_{j=k}^{n-1} \binom{n}{j+1} \binom{j}{k} a_{ad}.
\]
Eq. 30

27
Just as the second-order Eulerian number \( \left\langle \frac{n}{k} \right\rangle_{a \; d} \) was defined based on

\[
\begin{bmatrix} n \\ n-k \end{bmatrix}_{a \; d} = \sum_{j=1}^{k-1} \binom{k}{j} \binom{n+j}{2k},
\]

we define \( \left\langle \frac{n}{k} \right\rangle_{a \; d}^* \) based on

\[
\begin{bmatrix} n \\ n-k \end{bmatrix}_{a \; d}^* = \sum_{j=1}^{k-1} \binom{k}{j} \binom{n+k-j}{2k},
\]

where

\[
\begin{align*}
\left\langle \frac{0}{0} \right\rangle_{a \; d}^* &= 1, & \left\langle \frac{1}{0} \right\rangle_{a \; d}^* &= d - a, & \left\langle \frac{1}{1} \right\rangle_{a \; d}^* &= a, & \left\langle \frac{1}{2} \right\rangle_{a \; d}^* &= (d - a)(2d - a), \\
\left\langle \frac{2}{0} \right\rangle_{a \; d}^* &= -2a^2 + 3ad + d^2, & \left\langle \frac{2}{1} \right\rangle_{a \; d}^* &= a^2, & \left\langle \frac{2}{2} \right\rangle_{a \; d}^* &= (d - a)(2d - a)(3d - a), \\
\left\langle \frac{3}{0} \right\rangle_{a \; d}^* &= 3a^3 - 12a^2d + 7ad^2 + 8d^3, & \left\langle \frac{3}{1} \right\rangle_{a \; d}^* &= -3a^3 + 6a^2d + 4ad^2 + d^3, & \left\langle \frac{3}{2} \right\rangle_{a \; d}^* &= a^3
\end{align*}
\]

and in general

\[
\left\langle \frac{n}{k} \right\rangle_{a \; d}^* = [a + (n - k - 1)d] \left\langle \frac{n-1}{k-1} \right\rangle_{a \; d}^* + [-a + (n + k + 1)d] \left\langle \frac{n-1}{k} \right\rangle_{a \; d}^*.
\]

Note that

\[
\left\langle \frac{n}{n-k} \right\rangle_{a \; d}^* = (-1)^{k-1} \sum_{j=1}^{n-k-1} \binom{n-j}{k-1} \left\langle \frac{n}{j} \right\rangle_{a \; d}^*. \quad \text{Eq. 31}
\]

28
Based on any underlying infinite sequence \( A(i) \) in a commutative ring, we can construct the triangular array of numbers \( T(n, k \mid A(i) \mid u;v) \), with \( T(0, 0) = 1 \), \( T(n, 0) \) to be specified for each array and \( u, v \) each indicating which weight to be used among \( W(1) = 1, W(2) = A(n-1), W(3) = A(k), W(4) = A(n+k-1) + 2A(1) - A(2), W(5) = A(n-k+1), W(6) = A(k+2) - 2A(1), W(7) = A(2n-k), W(8) = (n+k-1)[A(2) - A(1)], W(9) = (k+2)A(1) - (k+1)A(2), W(10) = (n-k)[A(2) - A(1)] \) and \( W(11) = A(n+k+1) - 2A(1) \) in the recursive formula
\[
T(n, k) = W(u)T(n-1, k-1) + W(v)T(n-1, k),
\]
Eq. 32
where the index pair \((n, k)\) starts with \((0, 0)\) for Stirling arrays and \((0, -1)\) for Eulerian arrays.

In addition to the briefing given in ABSTRACT, we shall include more triangular arrays introduced in Section 4 as follows.

\[
\begin{bmatrix}
\binom{n}{k} \\
\end{bmatrix}_{a,d} = T(n, k+1 \mid A(i) = a + (i-1)d \mid 8;9),
\]

\[
T(n, 0) = [2A(1) - A(2)]T(n-1,0), n > 0;
\]

\[
\begin{bmatrix}
\binom{n}{k} \\
\end{bmatrix}_{a,d} = T(n, k+1 \mid A(i) = a + (i-1)d \mid 8;3),
\]

\[
T(n, 0) = [(n + 1)A(1) - nA(2)]T(n-1,0), n > 0;
\]

\[
\binom{n}{k}_{a,d} = T(n, k+1 \mid A(i) = a + (i-1)d \mid 5;10),
\]

\[
T(n, 0) = n[A(2) - A(1)]T(n-1,0), n > 0;
\]

\[
\binom{n}{k}_{a,d} = T(n, k+1 \mid A(i) = a + (i-1)d \mid 5;11),
\]

\[
T(n, 0) = [nA(2) - (n + 1)A(1)]T(n-1,0), n > 0.
\]
Let us elaborate by verifying $L_{2;3}(n,k) = \sum_{j=1}^{n} \binom{n}{j} \binom{j}{k}_{2;3}$ with Tables 10-12.

<table>
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</table>

Table 10. Table $\binom{n}{k}_{2;3}$ for illustrative purpose

<table>
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<th>$n \setminus k$</th>
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</table>

Table 11. Table $\binom{n}{k}_{2;3}$ for illustrative purpose

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<td>3640</td>
<td>780</td>
<td>52</td>
<td>1</td>
</tr>
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</table>

Table 12. Table $L_{2;3}(n,k)$ for illustrative purpose
\[ L_{2;3}(1,1) = \left[ \begin{array}{c} 1 \\ \frac{1}{2;3} \end{array} \right]_{2;3} \left[ \begin{array}{c} 1 \\ \frac{1}{2;3} \end{array} \right]_{2;3} = 1 \]

\[ L_{2;3}(2,1) = \left[ \begin{array}{c} 2 \\ \frac{1}{2;3} \end{array} \right]_{2;3} \left[ \begin{array}{c} 1 \\ \frac{1}{2;3} \end{array} \right]_{2;3} + \left[ \begin{array}{c} 2 \\ \frac{2}{2;3} \end{array} \right]_{2;3} \left[ \begin{array}{c} 2 \\ \frac{1}{2;3} \end{array} \right]_{2;3} = 4 \]

\[ L_{2;3}(2,2) = \left[ \begin{array}{c} 2 \\ \frac{2}{2;3} \end{array} \right]_{2;3} \left[ \begin{array}{c} 2 \\ \frac{2}{2;3} \end{array} \right]_{2;3} = 1 \]

\[ L_{2;3}(3,1) = \left[ \begin{array}{c} 3 \\ \frac{1}{2;3} \end{array} \right]_{2;3} \left[ \begin{array}{c} 1 \\ \frac{1}{2;3} \end{array} \right]_{2;3} + \left[ \begin{array}{c} 3 \\ \frac{2}{2;3} \end{array} \right]_{2;3} \left[ \begin{array}{c} 2 \\ \frac{1}{2;3} \end{array} \right]_{2;3} + \left[ \begin{array}{c} 3 \\ \frac{3}{2;3} \end{array} \right]_{2;3} \left[ \begin{array}{c} 3 \\ \frac{1}{2;3} \end{array} \right]_{2;3} = 28 \]

\[ L_{2;3}(3,2) = \left[ \begin{array}{c} 3 \\ \frac{2}{2;3} \end{array} \right]_{2;3} \left[ \begin{array}{c} 2 \\ \frac{2}{2;3} \end{array} \right]_{2;3} + \left[ \begin{array}{c} 3 \\ \frac{3}{2;3} \end{array} \right]_{2;3} \left[ \begin{array}{c} 3 \\ \frac{2}{2;3} \end{array} \right]_{2;3} = 14 \]

\[ L_{2;3}(3,3) = \left[ \begin{array}{c} 3 \\ \frac{3}{2;3} \end{array} \right]_{2;3} \left[ \begin{array}{c} 3 \\ \frac{3}{2;3} \end{array} \right]_{2;3} = 1 \]

\[ L_{2;3}(4,1) = \left[ \begin{array}{c} 4 \\ \frac{1}{2;3} \end{array} \right]_{2;3} \left[ \begin{array}{c} 1 \\ \frac{1}{2;3} \end{array} \right]_{2;3} + \left[ \begin{array}{c} 4 \\ \frac{2}{2;3} \end{array} \right]_{2;3} \left[ \begin{array}{c} 2 \\ \frac{1}{2;3} \end{array} \right]_{2;3} + \left[ \begin{array}{c} 4 \\ \frac{3}{2;3} \end{array} \right]_{2;3} \left[ \begin{array}{c} 3 \\ \frac{1}{2;3} \end{array} \right]_{2;3} + \left[ \begin{array}{c} 4 \\ \frac{4}{2;3} \end{array} \right]_{2;3} \left[ \begin{array}{c} 4 \\ \frac{1}{2;3} \end{array} \right]_{2;3} = 280 \]

\[ L_{2;3}(4,2) = \left[ \begin{array}{c} 4 \\ \frac{2}{2;3} \end{array} \right]_{2;3} \left[ \begin{array}{c} 2 \\ \frac{2}{2;3} \end{array} \right]_{2;3} + \left[ \begin{array}{c} 4 \\ \frac{3}{2;3} \end{array} \right]_{2;3} \left[ \begin{array}{c} 3 \\ \frac{2}{2;3} \end{array} \right]_{2;3} + \left[ \begin{array}{c} 4 \\ \frac{4}{2;3} \end{array} \right]_{2;3} \left[ \begin{array}{c} 4 \\ \frac{2}{2;3} \end{array} \right]_{2;3} = 210 \]

\[ L_{2;3}(4,3) = \left[ \begin{array}{c} 4 \\ \frac{3}{2;3} \end{array} \right]_{2;3} \left[ \begin{array}{c} 3 \\ \frac{3}{2;3} \end{array} \right]_{2;3} + \left[ \begin{array}{c} 4 \\ \frac{4}{2;3} \end{array} \right]_{2;3} \left[ \begin{array}{c} 4 \\ \frac{3}{2;3} \end{array} \right]_{2;3} = 30 \]

\[ L_{2;3}(4,4) = \left[ \begin{array}{c} 4 \\ \frac{4}{2;3} \end{array} \right]_{2;3} \left[ \begin{array}{c} 4 \\ \frac{4}{2;3} \end{array} \right]_{2;3} = 1 \]

Readers are urged to check \( L(n, k) = \sum_{j=1}^{n} \left[ \begin{array}{c} n \\ j \end{array} \right] \left[ \begin{array}{c} j \\ k \end{array} \right] \) by using Tables 3, 5 and 13.

<table>
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Table 13. Table L(n,k) for illustrative purpose
For more assurance, readers can verify \( L_{3:2}(n, k) = \sum_{j=1}^{n} \binom{n}{j} \binom{j}{k} \) with Tables 14-16.

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<td>1</td>
</tr>
</tbody>
</table>

Table 14. Table \([n]_{k, 3:2}\) for illustrative purpose

<table>
<thead>
<tr>
<th>( n \backslash k )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>9</td>
<td>8</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>27</td>
<td>49</td>
<td>15</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 15. Table \( \{n\}_{k, 3:2}\) for illustrative purpose

<table>
<thead>
<tr>
<th>( n \backslash k )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>48</td>
<td>16</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>480</td>
<td>240</td>
<td>30</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

Table 16. Table \( l_{3:2}(n, k) \) for illustrative purpose
Finally, we shall introduce an alternative

\[ W(12) = A(n+k-1) + A(1) - A(3) + A(2) \]

for

\[ W(4) = A(n+k-1) + A(1) - A(2) + A(1) \]

and point out that both

\[ \text{Lad}(n, k) = T(n, k \mid A(i) = a + (i-1)d \mid 1;12) \]

and

\[ \text{Lad}(n, k) = T(n, k \mid A(i) = a + (i-1)d \mid 1;4) \]

are valid, since

\[ A(1) - A(3) + A(2) = a - d = A(1) - A(2) + A(1), \]

which is not necessarily true for general \( A(i) \). Taking

\[ A(i) = C(i+1, 2) \]

for instance,

\[ L_{c2}(n, k) = T(n, k \mid A(i) = C(i+1, 2) \mid 1;12) \]

and

\[ L_{c2}(n, k) = T(n, k \mid A(i) = C(i+1, 2) \mid 1;4) \]

are different in values, since

\[ A(1) - A(3) + A(2) = C(1+1, 2) - C(3+1, 2) + C(2+1, 2) = -2 \]

while

\[ A(1) - A(2) + A(1) = C(1+1, 2) - C(2+1, 2) + C(1+1, 2) = -1. \]
Next, we consider \( \binom{n}{k} \) and \( \left\{ \binom{n}{k} \right\}^* \). We can use Eq. 32 for \( \left( \binom{r+1}{2} \right)_1 \), namely

\[
\left[ a \right] \left( \binom{n}{k} \right)^*_{1,2} = \left[ a \right] \left( \binom{n-1}{k-1} \right)_{1,2} + \left[ a \right] \left( \frac{n-1}{2} \right) \left( \binom{n-1}{k-1} \right)_{1,2}^*
\]

to tabulate Table 17.

<table>
<thead>
<tr>
<th>( n ) ( k )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>18</td>
<td>4</td>
<td></td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>180</td>
<td>288</td>
<td>127</td>
<td>20</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td>2700</td>
<td>4500</td>
<td>2193</td>
<td>427</td>
<td>35</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
<td>56700</td>
<td>97200</td>
<td>50553</td>
<td>11160</td>
<td>1162</td>
<td>56</td>
</tr>
</tbody>
</table>

Table 17. Table for \( \left[ a \right] \left( \binom{n}{k} \right)^*_{1,2} \)

By observing Table 17, we can see that

\[
\left[ a \right] \left( \binom{n}{k} \right)^*_{1,2} = \left( \binom{n}{r+1} \right), \quad \text{Eq. 33}
\]

\[
\left[ a \right] \left( \binom{n}{1} \right)^*_{1,2} = \prod_{t=1}^{n-1} \left( \binom{t}{r} \right), \quad \text{Eq. 34}
\]

and

\[
\left[ a \right] \left( \binom{n}{2} \right)^*_{1,2} = \prod_{t=1}^{n-1} \left( \binom{t}{r} \right) \sum_{j=1}^{n-1} \frac{1}{j^{r-1}}, \quad \text{Eq. 35}
\]
Then we use Eq. 32 for \( \binom{\binom{n}{2}}{k} \), namely \( \binom{n}{k} \left( \binom{\binom{n}{2}}{k} \right)^{\infty} = \binom{k+1}{2} \sum_{k=1}^{n-1} \binom{n-1}{k-1} \left( \binom{\binom{n}{2}}{k} \right)^{\infty} + \binom{n}{k} \left( \binom{\binom{n}{2}}{k} \right)^{\infty} \) to tabulate Table 18.

<table>
<thead>
<tr>
<th>n ( \setminus k )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>7</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>43</td>
<td>17</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>259</td>
<td>213</td>
<td>32</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>1555</td>
<td>2389</td>
<td>693</td>
<td>53</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>9331</td>
<td>25445</td>
<td>12784</td>
<td>1806</td>
<td>81</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td>55987</td>
<td>263781</td>
<td>217205</td>
<td>50710</td>
<td>4074</td>
<td>117</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 18. Table for \( \binom{n}{k} \left( \binom{\binom{n}{2}}{k} \right)^{\infty} \)

By observing the table above, we can derive

\[
\binom{n}{n-1} \left( \binom{\binom{n}{2}}{k} \right)^{\infty} = \binom{n+r-1}{r+1} \quad \text{Eq. 36}
\]

and

\[
\binom{n}{2} \left( \binom{\binom{n}{2}}{k} \right)^{\infty} = \frac{(r+1)^{n-1} - 1}{r} \quad \text{Eq. 37}
\]

Note that Eqs. 33-37 are the extensions of the following basic Stirling identities

\[
\binom{n}{n-1} = \binom{n}{n-1} = \binom{n}{2}, \quad \binom{n}{1} = (n-1)!, \quad \binom{n}{2} = (n-1)! \sum_{i=1}^{n-1} \frac{1}{i} \quad \text{and} \quad \binom{n}{2} = 2^{n-1} - 1.
\]
We shall now introduce the following method of common differences as special cases of diagonal (Stirling) polynomials $S(n, n-k)$, where $k = 2$. In APPENDIX B, Python produced tables will include the difference tables to facilitate the calculation of the coefficients of each polynomial in question as demonstrated in the following two examples.

As we can check in Table 17, the common difference for \( \binom{n}{n-2} \) is of degree 6. So by assuming

\[
\binom{n}{n-2} = \sum_{j=0}^{6} c_{n-j} n^j
\]

and taking \( n = 3, 4, 5, 6, 7, 8, 9 \) respectively, we can solve the following equations simultaneously.

\[
\begin{align*}
729c_6^6 + 243c_5^6 + 81c_4^6 + 27c_3^6 + 9c_2^6 + 3c_1^6 + c_0^6 &= 3 \\
4096c_6^6 + 1024c_5^6 + 256c_4^6 + 64c_3^6 + 16c_2^6 + 4c_1^6 + c_0^6 &= 27 \\
15625c_6^6 + 3125c_5^6 + 625c_4^6 + 125c_3^6 + 25c_2^6 + 5c_1^6 + c_0^6 &= 127 \\
46656c_6^6 + 7776c_5^6 + 1296c_4^6 + 216c_3^6 + 36c_2^6 + 6c_1^6 + c_0^6 &= 427 \\
117649c_6^6 + 16807c_5^6 + 2401c_4^6 + 343c_3^6 + 49c_2^6 + 7c_1^6 + c_0^6 &= 1162 \\
262144c_6^6 + 32768c_5^6 + 4096c_4^6 + 512c_3^6 + 64c_2^6 + 8c_1^6 + c_0^6 &= 2730 \\
53144c_6^6 + 65536c_5^6 + 729c_3^6 + 81c_2^6 + 9c_1^6 + c_0^6 &= 5754
\end{align*}
\]

or equivalently

36
\[3367c_6^6 + 78k_5^6 + 175c_4^6 + 37c_3^6 + 7c_2^6 + c_1^6 = 24\]
\[11529k_6^6 + 210k_5^6 + 369k_4^6 + 6k_3^6 + 9c_2^6 + c_1^6 = 100\]
\[3103k_6^6 + 465k_5^6 + 67k_4^6 + 9k_3^6 + 11c_2^6 + c_1^6 = 300\]
\[7093x_6^6 + 903k_5^6 + 1105c_4^6 + 127c_3^6 + 13c_2^6 + c_1^6 = 725\]
\[14449x_6^6 + 1596k_5^6 + 1695c_4^6 + 169c_3^6 + 15c_2^6 + c_1^6 = 1568\]
\[26929x_6^6 + 2465x_5^6 + 217x_4^6 + 17c_3^6 + c_2^6 + c_1^6 = 3024\]

or equivalently

\[8162c_6^6 + 1320x_5^6 + 194k_4^6 + 24c_3^6 + 2c_2^6 = 76\]
\[19502c_6^6 + 2550k_5^6 + 302x_4^6 + 30x_3^6 + 2c_2^6 = 200\]
\[39962c_6^6 + 4380x_5^6 + 434c_4^6 + 36c_3^6 + 2c_2^6 = 435\]
\[73502c_6^6 + 6930x_5^6 + 590x_4^6 + 42c_3^6 + 2c_2^6 = 833\]
\[124802c_6^6 + 10320x_5^6 + 770c_4^6 + 48c_3^6 + 2c_2^6 = 1456\]

or equivalently

\[11340c_6^6 + 1230x_5^6 + 108c_4^6 + 6c_3^6 = 124\]
\[20460c_6^6 + 1830x_5^6 + 132c_4^6 + 6c_3^6 = 235\]
\[33540c_6^6 + 2550x_5^6 + 156c_4^6 + 6c_3^6 = 398\]
\[51300c_6^6 + 3390x_5^6 + 180c_4^6 + 6c_3^6 = 623\]
or equivalently

\[ 9120 \alpha_6^6 + 600 \alpha_5^6 + 24 \alpha_4^6 = 111 \]
\[ 13080 \alpha_6^6 + 720 \alpha_5^6 + 24 \alpha_4^6 = 163 \]
\[ 17760 \alpha_6^6 + 840 \alpha_5^6 + 24 \alpha_4^6 = 225 \]

or equivalently

\[ 3960 \alpha_6^6 + 120 \alpha_5^6 = 52 \]
\[ 4680 \alpha_6^6 + 120 \alpha_5^6 = 62 \]

or equivalently \( 720 \alpha_6^6 = 10 \).

We can thus successively obtain

\[ \alpha_6^6 = \frac{1}{72}, \quad \alpha_5^6 = -\frac{1}{40}, \quad \alpha_4^6 = -\frac{1}{36}, \quad \alpha_3^6 = \frac{1}{24}, \quad \alpha_2^6 = \frac{1}{72}, \quad \alpha_1^6 = -\frac{1}{60} \quad \text{and} \quad \alpha_0^6 = 0. \]

Therefore, we have

\[
\begin{bmatrix}
\alpha_n \\
\alpha_{n-2} \\
\vdots
\end{bmatrix}
\left(\begin{array}{c}
\binom{n}{2} \\
\binom{n-2}{2} \\
\vdots
\end{array}\right)
= \frac{1}{72} n^6 - \frac{1}{40} n^5 - \frac{1}{36} n^4 + \frac{1}{24} n^3 + \frac{1}{72} n^2 - \frac{1}{60} n,
\]

which can further be converted into

\[
\begin{bmatrix}
\alpha_n \\
\alpha_{n-2} \\
\vdots
\end{bmatrix}
\left(\begin{array}{c}
\binom{n}{2} \\
\binom{n-2}{2} \\
\vdots
\end{array}\right)
= \frac{5n^2 + n - 3}{15} \binom{n+1}{4},
\]

Eq. 38

38
Similar for \( \binom{n}{n-2} \binom{(n+1)^2}{1} \), we can use Table 18 to set up the following equations simultaneously.

\[
729c_6^6 + 243c_5^6 + 81c_4^6 + 27c_3^6 + 9c_2^6 + 3c_1^6 + c_0^6 = 1
\]
\[
4096c_6^6 + 1024c_5^6 + 256c_4^6 + 64c_3^6 + 16c_2^6 + 4c_1^6 + c_0^6 = 43
\]
\[
15625c_6^6 + 3125c_5^6 + 625c_4^6 + 125c_3^6 + 25c_2^6 + 5c_1^6 + c_0^6 = 213
\]
\[
46656c_6^6 + 7776c_5^6 + 1296c_4^6 + 216c_3^6 + 36c_2^6 + 6c_1^6 + c_0^6 = 693
\]
\[
117649c_6^6 + 16807c_5^6 + 2401c_4^6 + 343c_3^6 + 49c_2^6 + 7c_1^6 + c_0^6 = 1806
\]
\[
262144c_6^6 + 32768c_5^6 + 4096c_4^6 + 512c_3^6 + 64c_2^6 + 8c_1^6 + c_0^6 = 4074
\]
\[
53144c_6^6 + 65536c_5^6 + 729c_4^6 + 81c_3^6 + 9c_2^6 + c_1^6 + c_0^6 = 8286
\]

and follow the same exact procedure as in the case of \( \binom{n}{n-2} \binom{(n+1)^2}{1} \) to obtain

\[
c_6^6 = \frac{1}{72}, \quad c_5^6 = \frac{1}{40}, \quad c_4^6 = -\frac{1}{36}, \quad c_3^6 = -\frac{13}{24}, \quad c_2^6 = \frac{1}{72}, \quad c_1^6 = \frac{31}{60} \quad \text{and} \quad c_0^6 = 0.
\]

Likewise, we can arrive at

\[
\binom{n}{n-2} \binom{(n+1)^2}{1}^n = \frac{5n^3 + 9n^2 - 5n - 186}{60} \binom{n+1}{3}. \quad \text{Eq. 39}
\]
6. EXPLORATION

To close out, let us consider Stirling numbers based on other sequences such as \((q^{-1})_i^n\) with \(q \neq 0\).

First, we look at \(\binom{n}{k}_{(q^{-1})_i^k}\). Based on \(\binom{1}{0}_{(q^{-1})_i^0} = 0\) and \(\binom{1}{1}_{(q^{-1})_i^1} = 1\), we can derive from Eq. 20 with \(u=1\) and \(v=2\) for \((q^{-1})_i^n\), namely

\[
\binom{n}{k}_{(q^{-1})_i^k} = \binom{n-1}{k}_{(q^{-1})_i^k} + q^{n-2} \binom{n-1}{k-1}_{(q^{-1})_i^k},
\]

the following:

\[
\binom{2}{1}_{(q^{-1})_i^1} = q^0 \binom{1}{1}_{(q^{-1})_i^1} = 1, \quad \binom{2}{2}_{(q^{-1})_i^2} = \binom{1}{1}_{(q^{-1})_i^1} = 1, \\
\binom{3}{1}_{(q^{-1})_i^1} = q^1 \binom{1}{1}_{(q^{-1})_i^1} = q, \quad \binom{3}{2}_{(q^{-1})_i^2} = \binom{2}{2}_{(q^{-1})_i^2} = q^2, \quad \binom{3}{3}_{(q^{-1})_i^3} = 1 + q, \\
\binom{4}{1}_{(q^{-1})_i^1} = q^2, \quad \binom{4}{2}_{(q^{-1})_i^2} = q^2 + q^3, \quad \binom{4}{3}_{(q^{-1})_i^3} = 1 + q + q^2, \quad \binom{4}{4}_{(q^{-1})_i^4} = 1, \ldots
\]

and in general

\[
\binom{n}{k}_{(q^{-1})_i^k} = q^{(n-k-1)} \prod_{i=1}^{k-1} \frac{1-q^{-k+i}}{1-q^i} = q^{(n-k-1)} \binom{n-1}{k}_{(q^{-1})_i^k},
\]

where \(\binom{n-1}{k}_{q}\) is known to be a \(q\) - Gaussian coefficient.
Likewise, we can use Eq. 32 with $u=1$ and $v=3$ for $\left(q^{j-1}\right)^n$, namely

\[
\binom{n}{k} \left(q^{j-1}\right)^n = \binom{k+1}{2} \binom{n-1}{k-1} \left(q^{j-1}\right)^n + \binom{n-1}{k} \left(q^{j-1}\right)^n
\]

to arrive at

\[
\binom{n}{k} \left(q^{j-1}\right)^n = \prod_{i=1}^{k-1} \frac{1-q^{-k+i}}{1-q^i} = \binom{n-1}{k}_q,
\]

Instead of finding more extensions along this line, let us use mathematical induction to prove

\[
\binom{n}{2} \left(q^{j-1}\right)_{\binom{\binom{\binom{\binom{3}{2}}}{n}}^n} = \binom{n}{2} + \binom{n}{3}
\]

as follows (showing only the inductive step):

\[
\binom{n}{2} \left(q^{j-1}\right)_{\binom{\binom{\binom{\binom{3}{2}}{2}}{n}}^n} = \binom{3}{2} \binom{n}{2} \left(q^{j-1}\right)_{\binom{\binom{\binom{3}{2}}{2}}{n}}^n + \binom{3}{1} \binom{n}{2} \left(q^{j-1}\right)_{\binom{\binom{3}{2}}{2}}^n
\]

\[
= 3\binom{n}{2} + 3\binom{n}{3} + 1
\]

\[
= 2\binom{n}{2} + \binom{n}{1} + 3\binom{n}{3} + \binom{n}{2}
\]

\[
= \binom{n+1}{2} + \binom{n+1}{3}.
\]

More triangular arrays can be constructed with appropriate pairs of weights in the global structures (recursive formulas) based on the local foundations (underlying sequences).
For example, we can use the following recursive formula

\[
\begin{align*}
\binom{n}{k}_{a,b,c} & = \binom{n-1}{k-1}_{a,b,c} + \binom{k}{0} a \binom{k-1}{1} b + \binom{k-2}{2} c \binom{n-1}{k}_{a,b,c}
\end{align*}
\]

Eq. 40

to obtain Table 19.

<table>
<thead>
<tr>
<th>n (\backslash) k</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>a</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>a^2</td>
<td>2a + b</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>a^3</td>
<td>3a^2 + 3ab + b^2</td>
<td>3a + 3b</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>a^4</td>
<td>\binom{5}{2}_{a,b}</td>
<td>\binom{5}{3}_{a,b}</td>
<td>\binom{5}{4}_{a,b} + c</td>
</tr>
</tbody>
</table>

Table 19. Table for Eq. 28

We can derive

\[
\begin{align*}
\binom{n}{4}_{a,b,c} & = \binom{n}{4}_{a,b} + c^{n-4} + \sum_{t=1}^{n-5} c^{n-4-t} (a + 3b)^{t-1} \binom{n}{t}_{a,b} + (n - 4 - t)(a + 3b)
\end{align*}
\]

and much more.

Let us now consider the ordered Bell polynomial for \( n > 1 \)

\[
E_n(a, d) = \sum_{k=1}^{n} \left( \sum_{j=1}^{k+1} \binom{k+1}{j} \right) \binom{n}{k}
\]

Eq. 41

and the Eulerian Bell polynomial

42
\[ E_n(a, d) = \sum_{k=0}^{n-1} 2^k \binom{n}{k}, \quad \text{for} \quad n > 1. \]  

Eq. 42

Can we generalize the well-known formula \( \sum_{k=0}^{n-1} 2^k \binom{n}{k} = \sum_{k=1}^{n} \left( \sum_{j=1}^{k+1} \binom{k+1}{j} \right) \binom{n}{k} \) to be AP based? We shall see that \( F_n(a, d) = E_n(a, d) \) only when \( a = d \). For our purpose, let us define the difference Bell polynomial \( D_n(a, d) \) to be

\[ D_n(a, d) = F_n(a, d) - E_n(a, d). \]  

Eq. 43

The Bell number \( B_n = \sum_{k=1}^{n} \binom{n}{k} \) satisfying \( B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k \) can be generalized to

\[ B_n(a, d) = \sum_{k=1}^{n} \left( \sum_{j=1}^{k+1} \binom{k+1}{j} \right) \binom{n}{k}, \]  

as follows.

\[ B_1(a, d) = 1, \]

\[ B_2(a, d) = a + d, \]

\[ B_3(a, d) = a^2 + 2ad + 2d^2, \]

\[ B_4(a, d) = a^3 + 3a^2d + 6ad^2 + 5d^3 \]

and

\[ B_5(a, d) = a^4 + 4a^3d + 12a^2d^2 + 20ad^3 + 15d^4 \]

so that
\[ B_1(a, d) = 1 = B_0, \]
\[ B_2(a, d) = aB_0 + dB_1, \]
\[ B_3(a, d) = a^2B_0 + 2adB_1 + d^2B_2, \]
\[ B_4(a, d) = a^3B_0 + 3a^2dB_1 + 3ad^2B_2 + d^3B_3 \]

and in general,
\[ B_{n+1}(a, d) = \sum_{k=0}^{n} a^{n-k} d^k \binom{n}{k} B_k. \]

Finally, we use Eq. 41 to find
\[ F_2(a, d) = ad + 2d^2, \]
\[ F_3(a, d) = a^2d + 4ad^2 + 8d^3, \]
\[ F_4(a, d) = a^3d + 6a^2d^2 + 24ad^3 + 44d^4, \]

......

and use Eq. 42 to find
\[ E_2(a, d) = 2ad + d^2, \]
\[ E_3(a, d) = a^3 + 6ad^2 + 6d^3, \]
\[ E_4(a, d) = 4a^3d + 6a^2d^2 + 28ad^3 + 37d^4, \]

......
In the same fashion, we can use Eq. 43 to obtain

\[ D_2(a,d) = (d-a)d \ , \]

\[ D_3(a,d) = (d-a)(a^2 + 2d^2) \]

\[ D_4(a,d) = (d-a)(3a^2d + 3ad^2 + 7d^3) \ . \]

\[ D_5(a,d) = (d-a)(a^4 + 12a^2d^2 + 24ad^3 + 38d^4) , \]

\[ D_6(a,d) = (d-a)(5a^4d + 10a^3d^2 + 70a^2d^3 + 185ad^4 + 271d^5) \ , \]

\[ D_7(a,d) = (d-a)(a^6 + 30a^4d^2 + 120a^3d^3 + 570a^2d^4 + 1620ad^5 + 2342d^6) , \]

……

In general, we propose the following

**CONJECTURE:**

\[ D_n(a,d) = (d-a)[(d-a)^{n-1} + E_{n-1}(a,d)] \ . \]

Eq. 44

In conclusion, let us get back to our original main goal and detail the method of differencing.

The polynomial expression \( \sum_{j=0}^{4} a_{4-j} n^{4-j} \) for \( \sum_{i=1}^{n} i^3 \) can be obtained by solving simultaneously

the following equations.
or equivalently

\[ c_4^4 + c_3^4 + c_2^4 + c_1^4 + c_0^4 = 1 \]
\[ 16c_4^4 + 8c_3^4 + 4c_2^4 + 2c_1^4 + c_0^4 = 9 \]
\[ 81c_4^4 + 27c_3^4 + 9c_2^4 + 3c_1^4 + c_0^4 = 36 \]
\[ 256c_4^4 + 64c_3^4 + 16c_2^4 + 4c_1^4 + c_0^4 = 100 \]
\[ 625c_4^4 + 125c_3^4 + 25c_2^4 + 5c_1^4 + c_0^4 = 225 \]

or equivalently

\[ c_4^4 + c_3^4 + c_2^4 + c_1^4 + c_0^4 = 1 \]
\[ 15c_4^4 + 7c_3^4 + 3c_2^4 + c_1^4 = 8 \]
\[ 50c_4^4 + 12c_3^4 + 2c_2^4 = 19 \]
\[ 60c_4^4 + 6c_3^4 = 18 \]
\[ 24c_4^4 = 6 . \]

The column coefficients of the latter are taken from the top diagonal of each of the difference table of the column coefficients of the former as in Table 20.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>1</th>
<th>1</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
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<td>15</td>
<td>81</td>
<td>65</td>
</tr>
<tr>
<td>81</td>
<td>65</td>
<td>50</td>
<td>8</td>
<td>7</td>
</tr>
<tr>
<td>81</td>
<td>65</td>
<td>50</td>
<td>8</td>
<td>7</td>
</tr>
<tr>
<td>256</td>
<td>175</td>
<td>110</td>
<td>27</td>
<td>19</td>
</tr>
<tr>
<td>256</td>
<td>175</td>
<td>110</td>
<td>27</td>
<td>19</td>
</tr>
<tr>
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<td>175</td>
<td>110</td>
<td>27</td>
<td>19</td>
</tr>
<tr>
<td>625</td>
<td>369</td>
<td>194</td>
<td>24</td>
<td>6</td>
</tr>
<tr>
<td>625</td>
<td>369</td>
<td>194</td>
<td>24</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>625</td>
<td>369</td>
<td>194</td>
<td>24</td>
</tr>
</tbody>
</table>

Table 20. The difference table for solving \( \sum_{i=1}^{n} t^3 \)
Let us rewrite the reduced simultaneous equations using the difference notations as

\[ c_4^4 + c_3^4 + c_2^4 + c_1^4 + c_0^4 = a_i \]

\[ \Delta n^4 c_4^4 + \Delta n^3 c_3^4 + \Delta n^2 c_2^4 + \Delta n c_1^4 = \Delta a \]

\[ \Delta^2 n^4 c_4^4 + \Delta^2 n^3 c_3^4 + \Delta^2 n^2 c_2^4 = \Delta^2 a \]

\[ \Delta^3 n^4 c_4^4 + \Delta^3 n^3 c_3^4 = \Delta^3 a \]

\[ \Delta^4 n^4 c_4^4 = \Delta^4 a , \]

where \( \Delta n^4 = 15 \), \( \Delta^2 n^4 = 50 \), \( \Delta^4 n^4 = 60 \), \( \Delta^4 n^4 = 24 \), \( \Delta^3 n^4 = 7 \), \( \Delta^5 n^4 = 12 \), \( \Delta^4 n^4 = 6 \).

\( \Delta n^2 = 3 \), \( \Delta^2 n^2 = 2 \), \( \Delta n = 1 \), \( a_1 = 1 \), \( \Delta a = 8 \), \( \Delta^2 a = 19 \), \( \Delta^3 a = 18 \) and \( \Delta^4 a = 6 \) so that

\[ c_4^4 = \frac{\Delta^4 a}{\Delta^4 n^4} = \frac{6}{24} = \frac{1}{4} ; \]

\[ c_3^4 = \frac{\Delta^3 a - \Delta^3 n^4 c_4^4}{\Delta^3 n^3} = \frac{18 - 60 \left(\frac{1}{4}\right)}{6} = \frac{1}{2} ; \]

\[ c_2^4 = \frac{\Delta^2 a - \Delta^2 n^4 c_4^4 - \Delta^2 n^3 c_3^4}{\Delta^2 n^2} = \frac{19 - 50 \left(\frac{1}{4}\right) - 12 \left(\frac{1}{2}\right)}{2} = \frac{1}{4} ; \]

\[ c_1^4 = \frac{\Delta a - \Delta n^4 c_4^4 - \Delta n^3 c_3^4 - \Delta n^2 c_2^4}{\Delta n} = \frac{8 - 15 \left(\frac{1}{4}\right) - 7 \left(\frac{1}{2}\right) - 3 \left(\frac{1}{4}\right)}{1} = 0 ; \]

\[ c_0^4 = 1 - c_4^4 - c_3^4 - c_2^4 - c_1^4 = 1 - \frac{1}{4} - \frac{1}{2} - \frac{1}{4} - 0 = 0 . \]

In general, \( c_k^k = \frac{\Delta^k a}{\Delta^k n^k} \), \( \Delta^k a = - \sum_{t=0}^{j-1} \Delta^{k-j} n^{k-t} c_k^k \), \( 1 \leq j \leq k - 1 \) and \( c_0^k = a_1 - \sum_{t=0}^{k-1} c_{k-t}^k \).
The polynomial coefficients for \( \sum_{i=1}^{n} i^2 \) can thus be obtained via Table 21.

\[
\begin{align*}
1 \\
5 & 4 \\
14 & 9 & 5 \\
30 & 16 & 7 & 2
\end{align*}
\]

Table 21. The difference table for solving \( \sum_{i=1}^{n} i^2 \)

\[
c_0^3 = \frac{\Delta^3 a}{\Delta^3 n} = \frac{2}{6} = \frac{1}{3};
\]

\[
c_2^3 = \frac{\Delta^2 a - 2 \Delta^2 n^3 c_3^3}{\Delta^2 n^3} = \frac{5 - 12 \left(\frac{1}{3}\right)}{6} = \frac{1}{2};
\]

\[
c_1^3 = \frac{\Delta a - \Delta n^3 c_3^3 - \Delta n^2 c_2^3}{\Delta n} = \frac{4 - 7 \left(\frac{1}{3}\right) - 3 \left(\frac{1}{2}\right)}{1} = \frac{1}{6};
\]

\[
c_0^3 = 1 - \frac{1}{3} - \frac{1}{2} - \frac{1}{6} = 0,
\]

while the polynomial expression for \( \sum_{i=1}^{n} i^4 \) via Table 22.

\[
\begin{align*}
1 \\
32 & 31 \\
243 & 211 & 180 & 17 & 16 \\
243 & 211 & 180 \\
1024 & 781 & 570 & 390 & 337 & 256 & 175 & 110 \\
3125 & 2101 & 1320 & 750 & 360 & 962 & 625 & 369 & 194 & 84 \\
7776 & 4651 & 2550 & 1320 & 480 & 120 & 2258 & 1296 & 617 & 302 & 108 & 24
\end{align*}
\]

Table 22. The difference table for solving \( \sum_{i=1}^{n} i^4 \)
The polynomial expressions for $f_i^n$, $i \geq 5$, can be obtained in the similar fashion.

Note that each Bernoulli number $c_i^k$ can now be calculated independently!

Readers are urged to consult (3)-(7) for further understanding and are also encouraged to prove Eqs. 30, 31 and even the unproven conjecture Eq. 44. In the end, we encourage readers make use of the programs provided in Section 7 to investigate more triangular arrays by trying more underlying base sequences and weights for the recursive formula (Eq. 32).
In conclusion, we shall prove the existence of general Stirling polynomials. Taking $\left[ \begin{array}{c} n \\ k \end{array} \right]_{a,d}$
as shown in Table 23 for instance, we first display difference Tables 24 and 25.

<table>
<thead>
<tr>
<th>S</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
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<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>a</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>(1,1,0)</td>
<td>2a+d</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>(1,3,2,0)</td>
<td>(3,6,2)</td>
<td>3a+d</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>(1,6,11,6,0)</td>
<td>(4,18,22,6)</td>
<td>(6,18,11)</td>
<td>4a+d</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>(1,10,35,50,24,0)</td>
<td>(5,40,105,100,24)</td>
<td>(10,60,105,50)</td>
<td>(10,40,35)</td>
<td>5a+d</td>
</tr>
</tbody>
</table>

Table 23. Table for $\left[ \begin{array}{c} n \\ k \end{array} \right]_{a,d}$

Note that (1,1,0) denote $1(a^2)+1(1d)+0(d^2)$, (1,3,2,0) denote $1(a^3)+3(a^2d)+2(ad^2)+0(d^3)$, …

<table>
<thead>
<tr>
<th>D1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>a</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2a+d</td>
<td>a+d</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>3a+3d</td>
<td>a+2d</td>
<td>d</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>4a+6d</td>
<td>a+3d</td>
<td>d</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>5a+10d</td>
<td>a+4d</td>
<td>d</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>6a+15d</td>
<td>a+5d</td>
<td>d</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>7a+21d</td>
<td>a+6d</td>
<td>d</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 24. Difference table for $\left[ \begin{array}{c} n \\ n-1 \end{array} \right]_{a,d}$

<table>
<thead>
<tr>
<th>D2</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(1,1,0)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>(3,6,2)</td>
<td>(2,5,2)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>(6,18,11)</td>
<td>(3,12,9)</td>
<td>(1,7,7)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>(10,40,35)</td>
<td>(4,22,24)</td>
<td>(1,10,15)</td>
<td>(0,3,8)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>(15,75,85)</td>
<td>(5,35,50)</td>
<td>(1,13,26)</td>
<td>(0,3,11)</td>
<td>(0,0,3)</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>(21,126,175)</td>
<td>(6,51,90)</td>
<td>(1,16,40)</td>
<td>(0,3,14)</td>
<td>(0,0,3)</td>
<td>(0,0,0)</td>
</tr>
</tbody>
</table>

Table 25. Difference table for $\left[ \begin{array}{c} n \\ n-2 \end{array} \right]_{a,d}$

We shall use mathematical induction to prove the following.
Theorem. Let $DD \left[ \frac{n}{n-t} \right]_{a:d}$ denote the degree of difference of $\left[ \frac{n}{n-t} \right]_{a:d}$. Then

$$DD \left[ \frac{n}{n-m-1} \right]_{a:d} = DD \left[ \frac{n}{n-m} \right]_{a:d} + 2.$$ 

Tables 24 and 25 established the basis of induction for proving the following when $m=1$,

since $DD \left[ \frac{n}{n-2} \right]_{a:d} = 5$ and $DD \left[ \frac{n}{n-1} \right]_{a:d} = 3$. As a matter of fact, we can rewrite Tables 24 and 25 into their symbolic forms as in Tables 26 and 27 to show the inductive step as follows.

<table>
<thead>
<tr>
<th>D1</th>
<th>0</th>
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<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>D1(1,0)</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>D1(2,0)</td>
<td>D1(2,1)</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>D1(3,0)</td>
<td>D1(3,1)</td>
<td>D1(3,2)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>D1(4,0)</td>
<td>D1(4,1)</td>
<td>D1(4,2)</td>
<td>D1(4,3)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>D1(5,0)</td>
<td>D1(5,1)</td>
<td>D1(5,2)</td>
<td>D1(5,3)</td>
<td>D1(5,4)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>D1(6,0)</td>
<td>D1(6,1)</td>
<td>D1(6,2)</td>
<td>D1(6,3)</td>
<td>D1(6,4)</td>
<td>D1(6,5)</td>
<td></td>
</tr>
<tr>
<td>7</td>
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<td>D1(7,1)</td>
<td>D1(7,2)</td>
<td>D1(7,3)</td>
<td>D1(7,4)</td>
<td>D1(7,5)</td>
<td>D1(7,6)</td>
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</table>

Table 26. Symbolic difference table for $\left[ \frac{n}{n-1} \right]_{a:d}$

<table>
<thead>
<tr>
<th>D2</th>
<th>0</th>
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<th>2</th>
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<th>4</th>
<th>5</th>
</tr>
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<tbody>
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<td>1</td>
<td>D2(1,0)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>D2(2,0)</td>
<td>D2(2,1)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>D2(3,0)</td>
<td>D2(3,1)</td>
<td>D2(3,2)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>D2(4,0)</td>
<td>D2(4,1)</td>
<td>D2(4,2)</td>
<td>D2(4,3)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>D2(5,0)</td>
<td>D2(5,1)</td>
<td>D2(5,2)</td>
<td>D2(5,3)</td>
<td>D2(5,4)</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>D2(6,0)</td>
<td>D2(6,1)</td>
<td>D2(6,2)</td>
<td>D2(6,3)</td>
<td>D2(6,4)</td>
<td>D2(6,5)</td>
</tr>
</tbody>
</table>

Table 27. Symbolic difference table for $\left[ \frac{n}{n-2} \right]_{a:d}$

Doing simple algebra from Table 24 to Table 25, we can further write Table 26 as
Let us derive the entries of Table 28. Since $D2(1,0) = a^2 + ad$ and $D2(2,0) = 3a^2 + 6ad + 2d^2$, then

$D2(2,1) = D2(2,0) - D2(1,0) = (a+2d)(2a+d) = D1(3,1)S(3,2)$

$D2(3,1) = D2(3,0) - D2(2,0) = (a+3d)(3a+3d) = D1(4,1)S(4,3)$

$D2(4,1) = D2(4,0) - D2(3,0) = (a+4d)(4a+6d) = D1(5,1)S(5,4)$

$D2(5,1) = D2(5,0) - D2(4,0) = (a+5d)(5a+10d) = D1(6,1)S(6,5)$

$D2(6,1) = D2(6,0) - D2(5,0) = (a+6d)(6a+15d) = D1(7,1)S(7,6)$

$D2(3,2) = D2(3,1) - D2(2,1) = (a+3d)(3a+3d) - (a+2d)(2a+d)$

$= (a+2d)((3a+3d) - (2a+d)) + d(3a+3d) = [D1(3,1)]^2 + dS(4,3)$

$D2(4,2) = D2(4,1) - D2(3,1) = (a+4d)(4a+6d) - (a+3d)(3a+3d)$

$= (a+3d)((4a+6d) - (3a+3d)) + d(4a+6d) = [D1(4,1)]^2 + dS(5,4)$

$D2(5,2) = D2(5,1) - D2(4,1) = (a+5d)(5a+10d) - (a+4d)(4a+6d)$

$= (a+4d)((5a+10d) - (4a+6d)) + d(5a+10d) = [D1(5,1)]^2 + dS(6,5)$

$D2(6,2) = D2(6,1) - D2(5,1) = (a+6d)(6a+15d) - (a+5d)(5a+10d)$

$= (a+5d)((6a+15d) - (4a+6d)) + d(6a+15d) = [D1(6,1)]^2 + dS(7,6)$

$D2(4,3) = D2(4,2) - D2(3,2) = \{[D1(4,1)]^2 + dS(5,4)\} - \{[D1(3,1)]^2 + dS(4,3)\}$

$= \{[D1(4,1)]^2 - [D1(3,1)]^2\} + [dS(5,4) - dS(4,3)] = d(2a+5d) + dD1(4,1)$
\[ D2(5,3) = D2(5,2) - D2(4,2) = \{[D1(5,1)]^2 + dS(6,5)\} - \{[D1(4,1)]^2 + dS(5,4)\} \]
\[ = \{[D1(5,1)]^2 - [D1(4,1)]^2 \} + [dS(6,5) - dS(5,4)] = d(2a+7d) + dD1(5,1) \]

\[ D2(6,3) = D2(6,2) - D2(5,2) = \{[D1(6,1)]^2 + dS(7,6)\} - \{[D1(5,1)]^2 + dS(6,5)\} \]
\[ = \{[D1(6,1)]^2 - [D1(5,1)]^2 \} + [dS(7,6) - dS(6,5)] = d(2a+9d) + dD1(6,1) \]

\[ D2(5,4) = D2(5,3) - D2(4,3) = [d(2a+7d) + dD1(5,1)] - [d(2a+5d) + dD1(4,1)] \]
\[ = 2d^2 + dD1(5,2) \]

\[ D2(6,4) = D2(6,3) - D2(5,3) = [d(2a+9d) + dD1(6,1)] - [d(2a+7d) + dD1(5,1)] \]
\[ = 2d^2 + dD1(6,2) \]

\[ D2(6,5) = (a+2d)D1(6,4) + dD1(6,3) \]

We can apply the above process on Table 29 to arrive at \( DD \left[ \frac{n}{n-3} \right]_{a,d} = 7 \) as follows.

<table>
<thead>
<tr>
<th>D3</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td>S(4,1)</td>
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<td></td>
<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>2</td>
<td>S(5,2)</td>
<td>D2(4,1)</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>S(6,3)</td>
<td>D2(5,1)</td>
<td>D2(5,2)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>S(7,4)</td>
<td>D2(6,1)</td>
<td>D2(6,2)</td>
<td>D2(6,3)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>S(8,5)</td>
<td>D2(7,1)</td>
<td>D2(7,2)</td>
<td>D2(7,3)</td>
<td>D2(7,4)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>S(9,6)</td>
<td>D2(8,1)</td>
<td>D2(8,2)</td>
<td>D2(8,3)</td>
<td>D2(8,4)</td>
<td>D2(8,5)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>S(10,7)</td>
<td>D2(9,1)</td>
<td>D2(9,2)</td>
<td>D2(9,3)</td>
<td>D2(9,4)</td>
<td>D2(9,5)</td>
<td>D2(9,6)</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>S(11,8)</td>
<td>D2(10,1)</td>
<td>D2(10,2)</td>
<td>D2(10,3)</td>
<td>D2(10,4)</td>
<td>D2(10,5)</td>
<td>D2(10,6)</td>
<td>D2(10,7)</td>
</tr>
</tbody>
</table>

Table 29. Symbolic difference table for \( \left[ \frac{n}{n-3} \right]_{a,d} \)

The exact same process can be used to obtain the entries of Table 30 as those in Table 28. We first extend from Table 28 that \( D2(7,4) = 2d^2 + dD1(7,2) \) and hence

\[ D2(7,5) = D2(7,4) - D2(6,4) = [2d^2+dD1(7,2)] - [2d^2+dD1(6,2)] = dD1(7,3), \]

which is equal to 0 as in Table 24 as well as \( D2(8,5), D2(8,6) \) so that \( D3(8,7) = 0 \) in Table 30.
The above process provided the inductive step for proving the Theorem, since we can successively obtain

\[ DD \left[ \frac{n}{n-1} \right]_{a,d} = 3, \]
\[ DD \left[ \frac{n}{n-2} \right]_{a,d} = 5, \]
\[ DD \left[ \frac{n}{n-3} \right]_{a,d} = 7, \]
\[ \ldots \]
\[ DD \left[ \frac{n}{n-m} \right]_{a,d} = 2m + 1, \]
\[ DD \left[ \frac{n}{n-m-1} \right]_{a,d} = 2m + 3, \]
\[ \ldots \]
As a matter of fact, the Theorem can be extended for any polynomial base $A(i)$. It suffices to look at $\{n\}_{(a+b(i-1)+c(i-1)^2)^{\infty}}^1$ as follows.

\[
D_1 \begin{array}{cccc}
1 & 2 & 3 & 4 \\
0 & a & a+b+c & a+2b+4c \\
1 & a+b+c & a+2b+4c & b+3c \\
2 & 3a+3b+5c & a+2b+4c & b+3c & 2c \\
3 & 4a+6b+14c & a+3b+9c & b+5c & 2c & 0 \\
4 & 5a+10b+30c & a+4b+16c & b+7c & 2c & 0 \\
5 & 6a+15b+55c & a+5b+29c & b+9c & 2c & 0 \\
\end{array}
\]

Table 31. Difference table for $\left\{\frac{n}{k}\right\}_{(a+b(i-1)+c(i-1)^2)^{\infty}}^1$

\[
D_2 \begin{array}{cccc}
1 & 2 & 3 & 4 \\
0 & (1,0,0,0,0) & (2,3,1,3,2,1) & (3,6,12,25,22,20) \\
1 & (1,0,0,0,0,0) & (2,3,1,3,2,1) & (3,6,12,25,22,20) \\
2 & (1,0,0,0,0,0) & (2,3,1,3,2,1) & (3,6,12,25,22,20) \\
3 & (1,0,0,0,0,0) & (2,3,1,3,2,1) & (3,6,12,25,22,20) \\
4 & (1,0,0,0,0,0) & (2,3,1,3,2,1) & (3,6,12,25,22,20) \\
5 & (1,0,0,0,0,0) & (2,3,1,3,2,1) & (3,6,12,25,22,20) \\
6 & (1,0,0,0,0,0) & (2,3,1,3,2,1) & (3,6,12,25,22,20) \\
7 & (1,0,0,0,0,0) & (2,3,1,3,2,1) & (3,6,12,25,22,20) \\
8 & (1,0,0,0,0,0) & (2,3,1,3,2,1) & (3,6,12,25,22,20) \\
\end{array}
\]

Table 32. Difference table for $\left\{\frac{n}{n-2}\right\}_{(a+b(i-1)+c(i-1)^2)^{\infty}}^1$

Note: $(1,1,1)$ denote $1(a)+1(b)+1(c)$, $(2,1,1)$ denote $2(a)+1(b)+1(c)$ and $(2,3,1,3,2,1)$ denote $2(a^2)+3(ab)+1(b^2)+3(ac)+2(bc)+1(c^2)$, …
Finally, by taking $a=1$ and $d=i = \sqrt{-1}$ in Eq. 32, we can derive from Tables 33 and 34 the following diagonal polynomials

$$\left[\frac{n}{n-1}\right]_{1;i} = n + \frac{n(n-1)}{2} i,$$

$$\left[\frac{n}{n-2}\right]_{1;i} = -\frac{n(n+1)(3n^2-n-14)}{24} + \frac{n^2(n+1)}{2} i,$$

$$\left\{\frac{n}{n-1}\right\}_{1;i} = n + \frac{n(n-1)}{2} i,$$

$$\left\{\frac{n}{n-2}\right\}_{1;i} = -\frac{n(n+1)(3n^2-5n-10)}{24} + \frac{n(n^2-1)}{2} i,$$

in the two-dimensional space. We can extend the idea to the $n$-dimensional space, by taking into account $i_1 = \sqrt{-1}$ instead of $i = \sqrt{-1}$.

<table>
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<tr>
<th>$n \backslash k$</th>
<th>1</th>
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<th>4</th>
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<th>6</th>
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<td>1</td>
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<td>-95+10i</td>
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<td>7</td>
<td>190-90i</td>
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<td>-221-252i</td>
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<td>8</td>
<td>730+1050i</td>
<td>2856-1854i</td>
<td>2715-2231i</td>
<td>-91-2520i</td>
<td>-840-420i</td>
<td>-114+126i</td>
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<td>9</td>
<td>-6620+6160i</td>
<td>18690+16284i</td>
<td>21888-14820i</td>
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<td>20098-8820i</td>
<td>-24600i</td>
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<td>175138+170568i</td>
<td>85256+151954i</td>
<td>-57916+1864i</td>
<td>8909-24600i</td>
<td>-3818-2908i</td>
<td>-470+308i</td>
<td>9+36i</td>
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Table 33.

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</tr>
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<td>5i</td>
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<td>-49-2240i</td>
<td>-1064-630i</td>
<td>-426+252i</td>
<td>8+28i</td>
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<td>9</td>
<td>1</td>
<td>16-15i</td>
<td>119-617i</td>
<td>4288+1877i</td>
<td>9975-3538i</td>
<td>2037-4690i</td>
<td>-2300-1890i</td>
<td>-416+252i</td>
<td>9+36i</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 34.
Furthermore, we can verify $L_{1;i}(n, k) = \sum_{j=1}^{n} \left[ \begin{array}{c} n \\ j \end{array} \right] \left\{ j \right\}_{1;i}$ for $n=4$ with Tables 33-35.

\[
\begin{array}{cccccc}
 n \backslash k & 1 & 2 & 3 & 4 & 5 \\
 1 & 1 & & & & \\
 2 & 2 & 1 & & & \\
 3 & 4+2i & 4+2i & 1 & & \\
 4 & 4+12i & 6+18i & 6+6i & 1 & \\
 5 & -28+36i & -56+72i & -12+60i & 8+12i & 1 \\
\end{array}
\]

Table 35. Table for $L_{1;i}(n, k)$

\[
L_{1;i}(1,1) = \left[ \begin{array}{c} 1 \\ 1 \end{array} \right]_{1;i} \left\{ 1 \right\}_{1;i} = 1
\]

\[
L_{1;i}(2,1) = \left[ \begin{array}{c} 2 \\ 1 \end{array} \right]_{1;i} \left\{ 1 \right\}_{1;i} + \left[ \begin{array}{c} 2 \\ 2 \end{array} \right]_{1;i} \left\{ 2 \right\}_{1;i} = 2
\]

\[
L_{1;i}(2,2) = \left[ \begin{array}{c} 2 \\ 2 \end{array} \right]_{1;i} \left\{ 2 \right\}_{1;i} = 1
\]

\[
L_{1;i}(3,1) = \left[ \begin{array}{c} 3 \\ 1 \end{array} \right]_{1;i} \left\{ 1 \right\}_{1;i} + \left[ \begin{array}{c} 3 \\ 2 \end{array} \right]_{1;i} \left\{ 2 \right\}_{1;i} + \left[ \begin{array}{c} 3 \\ 3 \end{array} \right]_{1;i} \left\{ 3 \right\}_{1;i} = 4 + 2i
\]

\[
L_{1;i}(3,2) = \left[ \begin{array}{c} 3 \\ 2 \end{array} \right]_{1;i} \left\{ 2 \right\}_{1;i} + \left[ \begin{array}{c} 3 \\ 3 \end{array} \right]_{1;i} \left\{ 3 \right\}_{1;i} = 4 + 2i
\]

\[
L_{1;i}(3,3) = \left[ \begin{array}{c} 3 \\ 3 \end{array} \right]_{1;i} \left\{ 3 \right\}_{1;i} = 1
\]

\[
L_{1;i}(4,1) = \left[ \begin{array}{c} 4 \\ 1 \end{array} \right]_{1;i} \left\{ 1 \right\}_{1;i} + \left[ \begin{array}{c} 4 \\ 2 \end{array} \right]_{1;i} \left\{ 2 \right\}_{1;i} + \left[ \begin{array}{c} 4 \\ 3 \end{array} \right]_{1;i} \left\{ 3 \right\}_{1;i} + \left[ \begin{array}{c} 4 \\ 4 \end{array} \right]_{1;i} \left\{ 4 \right\}_{1;i} = 4 + 12i
\]

\[
L_{1;i}(4,2) = \left[ \begin{array}{c} 4 \\ 2 \end{array} \right]_{1;i} \left\{ 2 \right\}_{1;i} + \left[ \begin{array}{c} 4 \\ 3 \end{array} \right]_{1;i} \left\{ 3 \right\}_{1;i} + \left[ \begin{array}{c} 4 \\ 4 \end{array} \right]_{1;i} \left\{ 4 \right\}_{1;i} = 6 + 18i
\]

\[
L_{1;i}(4,3) = \left[ \begin{array}{c} 4 \\ 3 \end{array} \right]_{1;i} \left\{ 3 \right\}_{1;i} + \left[ \begin{array}{c} 4 \\ 4 \end{array} \right]_{1;i} \left\{ 4 \right\}_{1;i} = 6 + 6i
\]

\[
L_{1;i}(4,4) = \left[ \begin{array}{c} 4 \\ 4 \end{array} \right]_{1;i} \left\{ 4 \right\}_{1;i} = 1
\]
A Python program is written to generate Pascal, Stirling, Lah and Euler triangles using their respective recursion formulas. A generalized triangle GT is constructed with \((n,k)\) index shifting. The input sequence can be categorized as (1) AP based (linear), (2) non-AP based (non-linear exponential) and (3) Discrete (mapping) functions. Both (1) and (2) are essentially the formulated versions of (3) which can be any random integers. Two Python files are shown here: Triangle_GT_OneWs_Ai.py which can be run with the python.exe command in an interactive mode. The second Python program Triangle_GT_OneWs_Ai_dataFiles.py is run with three data files in a batch mode. Each of the three data files AP_Cases.txt, APnot_Cases.txt and InputSeq_Cases.txt corresponds to the above three categories. The input data format for each category is shown below as:

1. AP: \(n, a, d, u, v\) where \(A(i)=a+(i-1)*d\)
2. nonAP: \(n, A, u, v\) where \(A = [0=>C(i+1,2), 1=>i**2, 2=>i**3, \text{ etc}]\)
3. InputSeq: \(n, u, v, \) input sequence, which is a finite series, depending on \(n\).

Most of the \((a,d)\) are in the single digits as shown below:
The n-th power of the formula is 2, 3 or 4 as shown below.
Program Triangle_GT_OneWs_Ai.py

```python
import numpy as np
import sys

class Pascal:
    # A(i) = a + (i - 1) * d; A(i)=i or Unity sequence when a=d=1
    # A() has to be an instance method, static without self is called by
    # Pascal.A(i),
    # but then self is not defined for self.a or self.d
    def A(self, i):
        # AP
        if i < 0:
            print(f'In A(i) with i = {i}')
            sys.exit('Exit due to i < 0 passed to A(i)'
        return self.a + (i - 1) * self.d

class PtInit:
    '''
    m = self.n + 1
    pt = np.zeros((m, m), dtype=np.int64)
    pt[0][0] = self.iv  # old way is pt[1][1]
    '''
    def __init__(self, n, iv, A):
        self.n = n
        self.A = A
        self.pt = np.zeros((self.n, self.n), dtype=np.int64)
        self.pt[0][0] = iv
        self.mapTn0 = {5, 6): self.tn0_567, (5, 10): self.tn0_510, (5, 11):
                        self.tn0_511,
        (8, 3): self.tn0_83, (8, 9): self.tn0_89}

def tn0_567(self):
    for i in range(1, self.n):
        self.pt[i][0] = (self.A(2) - 2 * self.A(1)) * self.pt[i - 1][0]
```

60
```python
def tn0_510(self):
    for i in range(1, self.n):
        self.pt[i][0] = i * (self.A(2) - self.A(1)) * self.pt[i - 1][0]

def tn0_511(self):
    for i in range(1, self.n):
        self.pt[i][0] = (i * self.A(2) - (i + 1) * self.A(1)) *
                       self.pt[i - 1][0]

def tn0_83(self):
    for i in range(1, self.n):
        self.pt[i][0] = ((i + 1) * self.A(1) - i * self.A(2)) *
                         self.pt[i - 1][0]

def tn0_89(self):
    for i in range(1, self.n):
        self.pt[i][0] = (2 * self.A(1) - self.A(2)) * self.pt[i - 1][0]

def w0(self, n, k):
    return 0

def w1(self, n, k):  # no translation as Stirling's
    return 1

def w2(self, n, k):
    return self.A(n - 1)

def w3(self, n, k):
    return self.A(k)

def w4(self, n, k):  # 6,1,1,4,s for Lah numbers
    return self.A(n + k - 1) + 2 * self.A(k) - self.A(2)

def w5(self, n, k):  # 6,1,1,5,6,e for 1st order Eulerian numbers
    return self.A(n - k + 1)

def w6(self, n, k):  # 6,1,1,5,6,e for 1st and 2nd order Eulerian numbers
    return self.A(k + 2) - 2 * self.A(k)

def w7(self, n, k):  # 6,1,1,7,6,e for 2nd order Eulerian numbers
    return self.A(2 * n - k)
```

def w8(self, n, k):
    return (n + k - 1) * (self.A(2) - self.A(1))

def w9(self, n, k):
    return (k + 2) * self.A(1) - (k + 1) * self.A(2)

def w10(self, n, k):
    return (n - k) * (self.A(2) - self.A(1))

def w11(self, n, k):
    return self.A(n + k + 1) - 2 * self.A(1)

def __init__(self, *niv, u=1, v=1, a=2, d=3):  # a, d defaults to 2, 3
    self.n, self.iv = niv
    self.a = a  # A(i) = a + (i - 1)d
    self.d = d
    self.u = u  # W(u) = A(f(n,k)) which is a weight function of A(i) as
    self.v = v  # W(v), u, v are used to index into W's
    type = 'Stirling' if self.u == 1 else 'Euler'
    print(f"Dim n = {self.n}, (a, d) = ({self.a}, {self.d}), (u, v) = {self.u}, {self.v}), type = {type}"
    )
    self.config()

def config(self):
    ptInit = Pascal.PtInit(self.n + 1, self.iv, self.A)
    uv = (self.u, self.v)
    if uv in ptInit.mapTn0.keys():
        ptInit.mapTn0[uv] =
    self.t = ptInit.pt
    self.w = [self.w0, self.w1, self.w2, self.w3, self.w4, self.w5, self.w6, self.w7, self.w8, self.w9,
        self.w10, self.w11]
    self.indexShift = lambda i, j: (i, j - 1) if self.type == 'e' else (i, j)  # This is the Dec-23-2020 working piece
    self.indexShift = lambda i, j: (i, j) if self.u == 1 else (i, j)  # W(n,k) shifted to the T index, so no work here.

def recursive(self, row, col):
    if row == 1 and col == 1:
return self.t[row][col]

n, k = self.indexShift(row, col)
result = self.w[self.u](n, k) * self.recursive(row - 1, col - 1) + 
self.w[self.v](n, k) * self.recursive(row - 1, col)
return result

# Calculate the binomial coefficient C(i,j) for the Pascal triangle.
# This applies to both Stirling and Euler.
def pascal(self):
    for i in range(1, self.n + 1): # old way is range(2,..)
        for j in range(1, i + 1): # diagonal inclusive
            # self.t[i][j] = self.recursive(i, j)
            self.t[i][j] = self.w[self.u](*self.indexShift(i, j)) * 
            self.t[i - 1, j - 1] + 
            self.w[self.v](*self.indexShift(i, j)) *

    return self.t

def difference(self, diag):
    m = self.n - diag
d = np.zeros((m, m), dtype=np.int64)
    for i in range(m):
        d[0,i] = self.t[diag+i+1][i+1] # initialize first row with {n,n-diag}'

    for i in range(1, m):
        for j in range(i, m):
            d[i,j] = d[i-1][j] - d[i-1][j-1] # Take a difference
    return d.transpose()

def lowerTrianger(self, array, **kwArgs):
    print("\t\t", f"##### Triangle array for dimension {kwArgs['end']-kwArgs['start']} ""#####")
    for i in range(kwArgs['start'], kwArgs['end']):
        for j in range(kwArgs['start'], i + 1):
            print("{:9d}".format(array[i][j]), end='\t')
        print()
pascal()}
```python
def diagonalDiff(self, diagonal=1):
    print(f"\t%%\n    Diagonal differences for S(n, n-{diagonal})
    %%%")
    d = self.difference(diagonal)
    self.lowerTrianger(d, start=0, end=self.n - diagonal)

class TriangleNAP(Pascal):
    # Non AP A(i) = a + (i-1)d
    def A0(self, i):
        return (i + 1) * i / 2  # A(i)=C(i+1,2)
    def A1(self, i):
        return i ** 2  # n*n diagonal converges but not n^n (n**n)
    def A2(self, i):
        return i ** 3
    def A3(self, i):
        return i ** 4
    def A4(self, i):
        return 2 ** i
    def A5(self, i):
        return 3 ** i
    def w12(self, n, k):
        return self.A(n * n)
    __init__(self, *niv, u=1, v=1, A=0):  # a defaults to 0 or A0
        self.n, self.iv = niv
        self.a = A
        self.u = u  # W(u) = A(f(n,k))
        self.v = v  # W(v), u, v are indexed to W which is a weight function of
        type = 'Stirling' if self.u == 1 else 'Euler'
        print(f"Dim n = {self.n}, A = {self.a}, (u, v) = ({self.u}, {self.v}),
        type = {type}")
        self.A = nonAp[self.a]
        self.config()
```

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class AinputSeq(Pascal):

def A(self, i):
    # input sequence
    if i >= self.f.__len__():
        print(f"In AinputSeq.A(i) with the pass-in i = {i} >= ips length {len(self.f)}")
        sys.exit('Exit due to AinputSeq.A(i) > ips[] length, consider a longer ips or a smaller n!!')
    return self.f[i-1] if self.u == 1 else self.f[i]

def __init__(self, *niv, **kwargs):
    self.n, self.iv = niv
    self.u = kwargs['u']
    self.v = kwargs['v']
    self.f = kwargs['ips']
    type = 'Stirling' if self.u == 1 else 'Euler'
    print(f"Dim n = {self.n}, (u, v) = ({self.u}, {self.v}), ips = {self.f}, type = '{type}"")
    self.config()

if __name__ == '__main__':  # Interactive mode
    ap = int(input(f"Enter 0 for linear A(i)=a+(i-1)d, 1 for non-linear A(i) and 2 for input sequence : "))
    print("u = 1 for Stirling, else for Euler")
    loop_max = 10
    for loop in range(loop_max):
        print(f"loop time (max. = {loop_max}) = {loop + 1}\n")
        if ap == 0:
            lst = [item for item in input(f"Enter n, a, d, u, v (u=1 => Stirling, else Euler) : ").split(',')]  # Stirling, else Euler):
            lin = [int(x) for x in lst[:]]
            for c in (Pascal(*{lin[0], 1}, **{'a': lin[1], 'd': lin[2], 'u': lin[3], 'v': lin[4]}),):
                pt = c.pascal()
                # print(f'Pascal size = {pt.size}, shape = {pt.shape} 2-D array pt =\n\n{pt}')
        elif ap == 1:
lst = [item for item in input(f"Enter n, A
[0=>C(i+1,2), 1=>i**2, 2=>i**3], u, v (u=1 => Stirling, else Euler) : ").split(',')]  
lin = [int(x) for x in lst[:]]
for c in (TriangleNAP(*([lin[0], 1]), **{'A': lin[1], 'u': lin[2], 'v': lin[3]})),):
    pt = c.pascal()
    # print(f'Pascal size = {pt.size}, shape = {pt.shape} 2-D array pt =>

    elif ap == 2:
        lst = [item for item in input(f"Enter n, u, v, input sequence : ").split(',')]  
        lin = [int(x) for x in lst[:]]
        for c in (AinputSeq(*([lin[0], 1]), **{'u': lin[1], 'v': lin[2], 'ips': lin[3]})),):
            pt = c.pascal()
            # print(f'Pascal size = {pt.size}, shape = {pt.shape} 2-D array pt =>

        c.lowerTrianger(pt, start=0, end=c.n + 1)  # old way start=1, changed to =1 to account for T(n,0) != 0
        if c.u == 1:
            diag = int(input("Please enter the m-th diagonal for S(n, n-m)
differences: "))
            c.diagonalDiff(diag)
Program Triangle_GT_OneWs_Ai_dataFiles.py

```python
from Triangle_GT_OneWs_Ai import *

fd = {0: 'AP_Cases.txt', 1: 'APnot_Cases.txt', 2: 'InputSeq_Cases.txt'}
diag = {0: 2, 1: 1, 2: 1}
for i in range(3):
    loop = 0
    apFile = open(fd[i], 'r')
    for line in apFile:
        loop += 1
        print(f"\t***** Triangle case # (loop) = {line.rstrip()} *****")
        lst = line.rstrip().split(',')
        lin = [int(x) for x in lst[:]]
        if i == 0:
            c = Pascal(*([0], 1), **{'a': lin[1], 'd': lin[2], 'u': lin[3], 'v': lin[4]})
        elif i == 1:
            c = TriangleNAP(*([0], 1), **{'A': lin[1], 'u': lin[2], 'v': lin[3]})
        elif i == 2:
            c = AinputSeq(*([0], 1), **{'u': lin[1], 'v': lin[2], 'ips': lin[3:]})
        pt = c.pascal()
        # print(f'Pascal size = (pt.size), shape = (pt.shape) 2-D array
        pt =>\n        c.lowerTrianger(pt, start=0, end=c.n + 1)
        if c.u == 1:
            c.diagonalDiff(diag[i])
        print(f"Total readline of {apFile.name} = {loop}\n")
apFile.close()
```

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**GLOSSARY**

**Polynomial:** A mathematical expression such as $ax^3+bx^2-cx$, where $x$ is a variable and $a$, $b$, $c$ are called coefficients.

**Binomial expansion:** According to the binomial theorem, it is possible to expand the polynomial $(x + y)^n$ into a sum involving terms of the form $a\, x^b\, y^c$, where the exponents $b$ and $c$ are nonnegative integers with $b + c = n$, and the coefficient $a$ of each term is a specific positive integer depending on $n$ and $b$.

**Combinatorics:** The branch of mathematics dealing with combinations of objects belonging to a finite set in accordance with certain constraints.

**Mathematical induction:** To prove a statement $S(n)$ is true for any natural number $n$, it suffices first to establish the inductive basis [to prove $S(1)$ is true] and then to provide the inductive step [to prove $S(m+1)$ is true by assuming $S(m)$ is true].

**Python:** An interpreted, high-level and general-purpose programming language used worldwide. Python’s design philosophy emphasizes code readability with its notable use of significant whitespace.
REFERENCES


APPENDIX A: LIST OF TABLES

Table 1. Pascal Triangle
Table 2. Bernoulli triangle
Table 3. Table for Stirling numbers of the first kind
Table 4. Table for general Stirling numbers of the second kind \( \binom{n}{k}_{2,3} \)
Table 5. Table for Stirling numbers of the second kind
Table 6. Table for general Stirling numbers of the first kind \( \left[ \begin{array}{c} n \\ k \end{array} \right]_{a,d} \)
Table 7. Table for general Stirling numbers of the second kind \( \left[ \begin{array}{c} n \\ k \end{array} \right]_{a,d} \)
Table 8. Table for general first order Eulerian numbers \( \left\langle \left\langle n \right\rangle \right\rangle_{a,d} \)
Table 9. Table for general second order Eulerian numbers \( \left\langle \left\langle \left\langle n \right\rangle \right\rangle \right\rangle_{a,d} \)
Table 10. Table \( \left[ \begin{array}{c} n \\ k \end{array} \right]_{2,3} \) for illustrative purpose
Table 11. Table \( \left\{ \begin{array}{c} n \\ k \end{array} \right\}_{2,3} \) for illustrative purpose
Table 12. Table \( l_{2;3}(n,k) \) for illustrative purpose
Table 13. Table L(n,k) for illustrative purpose
Table 14. Table \( \left[ \begin{array}{c} n \\ k \end{array} \right]_{3,2} \) for illustrative purpose
Table 15. Table \( \left[ \begin{array}{c} n \\ k \end{array} \right]_{3,2} \) for illustrative purpose
Table 16. Table \( l_{3;2}(n,k) \) for illustrative purpose
Table 17. Table for $\binom{n}{k} (\binom{i}{2})^n$

Table 18. Table for $\binom{n}{k} (\binom{i}{2})^n$

Table 19. Table for Eq. 28

Table 20. The difference table for solving $\sum_{i=1}^{n} i^3$

Table 21. The difference table for solving $\sum_{i=1}^{n} i^2$

Table 22. The difference table for solving $\sum_{i=1}^{n} i^4$

Table 23. Table for $\binom{n}{k}_{a:d}$

Table 24. Difference table for $\binom{n}{n-1}_{a:d}$

Table 25. Difference table for $\binom{n}{n-2}_{a:d}$

Table 26. Symbolic difference table for $\binom{n}{n-1}_{a:d}$

Table 27. Symbolic difference table for $\binom{n}{n-2}_{a:d}$

Table 28. Calculation table for Table 27

Table 29. Symbolic difference table for $\binom{n}{n-3}_{a:d}$

Table 30. Calculation table for Table 29

Table 31. Difference table for $\binom{n}{n-1} (a+b(i-1)+c(i-1)^2)^\infty_1$

Table 32. Difference table for $\binom{n}{n-2} (a+b(i-1)+c(i-1)^2)^\infty_1$

Table 33. Table for $\binom{n}{k}_{1:i}$

Table 34. Table for $\binom{n}{k}_{1:i}$

Table 35. Table for $L_{1;i}(n, k)$
APPENDIX B: PYTHON PRODUCED TABLES

Sample data files are shown below:

**AP_Cases.txt**
10,1,1,1,2
10,2,3,1,2
10,1,1,1,3
10,2,3,1,3
10,1,1,1,4
10,2,3,1,4
10,1,1,5,6
10,2,3,5,6
10,1,1,7,6
10,2,3,7,6
10,1,1,8,3
10,2,3,8,3
10,1,1,8,9
10,2,3,8,9
10,1,1,5,10
10,2,3,5,10
10,1,1,5,11
10,2,3,5,11

**APnot_Cases.txt**
8,0,1,2
8,0,1,3
8,0,1,4
8,1,1,2
8,1,1,3
8,1,1,4
8,2,1,2
8,2,1,3
8,2,1,4
8,3,1,2
8,3,1,3
8,4,1,2
8,5,1,2
InputSeq_Cases.txt
6,1,2,1,3,6,10,15,21
5,1,2,-3,-2,-1,0,1,2,3
6,1,3,2,5,8,11,14,17,20
7,1,3,1,2,4,8,16,32,64,128,256
5,1,4,-2,0,2,4,6,8,10,12
7,1,4,2,6,12,20,30,42,56,72,90,110,132,156,182,210

Output from running Triangle_GT_OneWs_Ai_dataFiles.py :
>> python.exe PascalTriangle/Triangle_GT_OneWs_Ai_dataFiles.py

***** Triangle case # 1 = 10,1,1,1,2 *****
Dim n = 10, (a, d) = (1, 1), (u, v) = (1, 2), type = Stirling

#### Triangle array for dimension 11 ####

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45

%%%% Diagonal differences for S(n, n-2) %%%%

#### Triangle array for dimension 8 ####

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***** Triangle case # 2 = 10,2,3,1,2 *****
Dim n = 10, (a, d) = (2, 3), (u, v) = (1, 2), type = Stirling

#### Triangle array for dimension 11 ####

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#### Diagonal differences for S(n, n-2) ####

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***** Triangle case # 3 = 10,1,1,3 *****

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#### Triangle array for dimension 11 ####

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%%%   Diagonal differences for S(n, n-2)  %%%
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65  40  22  10
140 75  35  13  3
266 126 51  16  3  0
462 196 70  19  3  0  0
750 288 92  22  3  0  0  0

*****   Triangle case # 4 = 10,2,3,1,3   *****
Dim n = 10, (a, d) = (2, 3), (u, v) = (1, 3), type = Stirling
#####   Triangle array for dimension 11   #####
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0   2   1
0   4   7   1
0   8   39  15  1
0  16  203  159  26  1
0  32  1031  1475  445  40  1
0  64  5187  12831  6370 1005  57  1
0 128 25999 107835  82901 20440 1974  77  1
0 256 130123 888679 1019746 369061 53998 3514 100
1
0  512  650871  7239555 12105885 6186600 1287027 124278 5814
126  1
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%%%   Diagonal differences for S(n, n-2)  %%%
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***** Triangle case # 5 = 10,1,1,4 *****
Dim n = 10, (a, d) = (1, 1), (u, v) = (1, 4), type = Stirling

##### Triangle array for dimension 11 #####

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%%%%% Diagonal differences for S(n, n-2) %%%%%

##### Triangle array for dimension 8 #####

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for $S(n, n-2)$

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252 | 1

for $S(n, n-2)$

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***** Triangle case # 8 = 10,2,3,5,6 *****
Dim n = 10, (a, d) = (2, 3), (u, v) = (5, 6), type = Euler

#### Triangle array for dimension 11 ####

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***** Triangle case # 9 = 10,1,1,7,6 *****
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#### Triangle array for dimension 11 ####

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***** Triangle case # 10 = 10,2,3,7,6 *****
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#### Triangle array for dimension 11 ####

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****** Triangle case # 11 = 10,1,1,8,3 ******

Dim n = 10, (a, d) = (1, 1), (u, v) = (8, 3), type = Euler

##### Triangle array for dimension 11 #####

1

0 1

0 1 3

0 1 10 15

0 1 25 105 105

0 1 56 490 1260 945

0 1 119 1918 9450 17325 10395

0 1 246 6825 56980 190575 270270 135135

0 1 501 22935 302995 1636635 4099095 4729725 2027025

0 1 1012 74316 1487200 12122110 47507460 94594500 91891800

34459425

0 1 2035 235092 6914908 81431350 466876410 1422280860 2343240900

1964187225 |  654729075 |

****** Triangle case # 12 = 10,2,3,8,3 ******

Dim n = 10, (a, d) = (2, 3), (u, v) = (8, 3), type = Euler

##### Triangle array for dimension 11 #####

1

-1 3

4 0 27

-28 36 135 405

280 -264 1215 5670 8505

-3640 3672 1323 70875 229635 229635

58240 -58176 83727 598752 4439610 10103940 7577955
-1106560 1106688 -977589 7050645 66798270 287962290 492567075 295540245 
-24344320 -24344064 24992631 7050645 66798270 287962290 492567075 295540245 
-608608000 608608512 -605358765 1041376743 11616764445 127837574865 603591693705 19604169585 
1582618011975 678264862275 17041024000 -17041022976 13461901596 168398101872 2277630154800 16013749661910 
34459425 6086080000 608608512 -605358765 1041376743 11616764445 127837574865 603591693705 19604169585 
1582618011975 678264862275 17041024000 -17041022976 13461901596 168398101872 2277630154800 16013749661910 
57255602744340 10852237796400 103096259065800 38661097149675 

***** Triangle case # 13 = 10,1,1,8,9 *****
Dim n = 10, (a, d) = (1, 1), (u, v) = (8, 9), type = Euler

##### Triangle array for dimension 11 #####
1
0 1
0 -1 3
0 1 -10 15
0 -1 25 -105 105
0 1 -56 490 -1260 945
0 -1 119 -1918 9450 -17325 10395
0 1 -246 6825 -56980 190575 -270270 135135
0 -1 501 -22935 302995 -1636635 409909 2027025
0 1 -1012 74316 -1487200 12122110 -47507460 94594500 -91891800
34459425
0 -1 2035 -235092 6914908 -81431350 466876410 -1422280860 2343240900 -1964187225 654729075

***** Triangle case # 14 = 10,2,3,8,9 *****
Dim n = 10, (a, d) = (2, 3), (u, v) = (8, 9), type = Euler

##### Triangle array for dimension 11 #####
1
-1 3
1 -18 27
-1 81 -405 405
1 -336 4050 -11340 8505
-1 1359 -34398 198450 -382725 229635
1 -5454 269325 -2810052 10333575 -15155910 7577955
-1 21837 -2016171 35372295 -218638035 583502535 -689593909 295540245
1 -87372 14702796 -414208080 4009580190 -17207009820 358853783060 -35464829400 13299311025
-1 349515 -105540732 4627273068 -67036033350 431685784530 -1404013190580 2393875984500 -2034794586825 678264862275
***** Triangle case # 15 = 10,1,1,5,10 *****

Dim n = 10, (a, d) = (1, 1), (u, v) = (5, 10), type = Euler

##### Triangle array for dimension 11 #####

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***** Triangle case # 16 = 10,2,3,5,10 *****

Dim n = 10, (a, d) = (2, 3), (u, v) = (5, 10), type = Euler

##### Triangle array for dimension 11 #####

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***** Triangle case # 17 = 10,1,1,5,11 *****

Dim n = 10, (a, d) = (1, 1), (u, v) = (5, 11), type = Euler

##### Triangle array for dimension 11 #####

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***** Triangle case # 18 = 10,1,1,5,11 *****

Dim n = 10, (a, d) = (1, 1), (u, v) = (5, 11), type = Euler
#### Triangle array for dimension 11 ####

1
0 1
0 2 1
0 6 8 1
0 24 58 22 1
0 120 444 328 52 1
0 720 3708 4400 1452 114 1
0 5040 33984 58140 32120 5610 240 1
0 40320 341136 785304 644020 19950 494 1
0 362880 3733920 11026296 12440064 5765500 1062500 67260 1004

1
0 3628800 44339040 162186912 238904904 155357384 44765000 5326160 218848
2026 1

***** Triangle case # 18 = 10,2,3,5,11 *****

Dim n = 10, (a, d) = (2, 3), (u, v) = (5, 11), type = Euler

#### Triangle array for dimension 11 ####

1
1
2
4 19 4
28 222 147 8
280 3194 4128 887 16
3640 55024 113566 52538 4835 32
58240 1107336 3268788 2562676 555684 25167 64
1106560 25526192 100544412 117517960 45415640 5301150 128203 128
24344320 663605680 3325767376 5352311764 3189383200 695714590 47537320 646519 256
608608000 19213911360 118361719296 248493947496 208996478388 72479948400 9696965250 410038434
3245139 512
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146079679100 126758501016 267586181336 3449054700 16253855 1024

Total readline of AP_Cases.txt = 18

***** Triangle case # 1 = 8,0,1,2 *****

Dim n = 8, A = 0, (u, v) = (1, 2), type = Stirling

#### Triangle array for dimension 9 ####

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0 1
Diagonal differences for S(n, n-1)

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Triangle array for dimension 7

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Triangle case # 2 = 8,0,1,3

Dim n = 8, A = 0, (u, v) = (1, 3), type = Stirling

Diagonal differences for S(n, n-1)

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Triangle array for dimension 9

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Diagonal differences for S(n, n-1)
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Dim n = 8, A = 1, (u, v) = (1, 2), type = Stirling

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%%%%% Diagonal differences for S(n, n-1) %%%%%

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Dim n = 8, A = 1, (u, v) = (1, 2), type = Stirling

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%%%%% Diagonal differences for S(n, n-1) %%%%%
### Triangle array for dimension 7

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### Triangle case # 5 = 8,1,1,3

Dim n = 8, A = 1, (u, v) = (1, 3), type = Stirling

### Triangle array for dimension 9

%%% %%% %%% Diagonal differences for S(n, n-1) %%% %%% %%%

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### Triangle case # 6 = 8,1,1,4

Dim n = 8, A = 1, (u, v) = (1, 4), type = Stirling

### Triangle array for dimension 9
Diagonal differences for $S(n, n-1)$

Triangle array for dimension 7

2
16  14
50  34  20
112  62  28  8
210  98  36  8  0
352  142  44  8  0  0
546  194  52  8  0  0  0

Triangle case # 7 = 8, 2, 1, 2

$\text{Dim } n = 8, A = 2, (u, v) = (1, 2), \text{ type } = \text{ Stirling}$

Triangle array for dimension 9

1
0  1
0  1  1
0  8  9  1
0  216  251  36  1
0  13824  16280  2555  100  1
0  1728000  2048824  335655  15055  225  1
0  373248000  444273984  74550304  3587535  63655  441  1
0  128024064000  152759224512  26015028256  1305074809  25421200  214918  784

1

Diagonal differences for $S(n, n-1)$

Triangle array for dimension 7

1
9  8
36  27  19
100  64  37  18
225  125  61  24  6
441  216  91  30  6  0
784  343  127  36  6  0  0
***** Triangle case # 8 = 8,2,1,3 *****
Dim n = 8, A = 2, (u, v) = (1, 3), type = Stirling

##### Triangle array for dimension 9 #####

1
0 1
0 1 1
0 1 9 1
0 1 73 36 1
0 1 585 1045 100 1
0 1 4681 28800 7445 225 1
0 1 37449 782281 505280 35570 441 1
0 1 299593 21159036 33120201 4951530 130826 784 1

%%%%% Diagonal differences for S(n, n-1) %%%%%

##### Triangle array for dimension 7 #####

1
9 8
36 27 19
100 64 37 18
225 125 61 24 6
441 216 91 30 6 0
784 343 127 36 6 0 0

***** Triangle case # 9 = 8,2,1,4 *****
Dim n = 8, A = 2, (u, v) = (1, 4), type = Stirling

##### Triangle array for dimension 9 #####

1
0 1
0 2 1
0 42 60 1
0 2436 7182 270 1
0 289884 1510656 98172 776 1
0 60875640 5093890956 51185688 659220 1770 1
0 20515090680 257807639376 375166338380 706450368 3004470 3492 1
0 10380635884080 186415438359528 37549341219096 973563370980 5880147708 10655442

6230 1

%%%%% Diagonal differences for S(n, n-1) %%%%%
**** Triangle array for dimension 7 ****

2
60  58
270 210  152
776 506  296  144
1770 994  488  192  48
3492 1722  728  240  48  0
6230 2738 1016  288  48  0  0

**** Triangle case # 10 = 8,3,1,2 ****
Dim n = 8, A = 3, (u, v) = (1, 2), type = Stirling

**** Triangle array for dimension 9 ****

1
0  1
0  1  1
0 16  17  1
0 1296 1393  98  1
0 331776 357904 26481  354  1
0 20736000 22402176 16908529 247731  979  1
0 268738560000 29053981696 22137475360 337967905 1516515 2275  1
0 645241282560000 697854274212096 53442617921056 833598415265 3979120420 6978790 4676 1

%%% Diagonal differences for S(n, n-1) %%%

**** Triangle array for dimension 7 ****

1
17  16
98  81  65
354 256  175  110
979 625  369  194  84
2275 1296  671  302  108  24
4676 2401 1105  434  132  24  0

**** Triangle case # 11 = 8,3,1,3 ****
Dim n = 8, A = 3, (u, v) = (1, 3), type = Stirling

**** Triangle array for dimension 9 ****

1
0  1
0 1  1
0 1 17 1
0 1 273 98 1
0 1 4369 8211 354 1
0 1 69905 669460 98835 979 1
0 1 1118481 54296165 25971220 710710 2275 1
0 1 17895697 4399107846 6702928485 470164970 3659110 4676 1

Diagonal differences for S(n, n - 1)

Triangle array for dimension 7

1
17 16
98 81 65
354 256 175 110
979 625 369 194 84
2275 1296 671 302 108 24
4676 2401 1105 434 132 24 0

Triangle case # 12 = 8, 4, 1, 2

Dim n = 8, A = 4, (u, v) = (1, 2), type = Stirling

Triangle array for dimension 9

1
0 1
0 2 1
0 8 6 1
0 64 56 14 1
0 1024 960 280 30 1
0 32768 31744 9920 1240 62 1
0 2097152 2064384 666624 89280 5208 126 1
0 268435456 266338304 87392256 12094464 755904 21336 254 1

Diagonal differences for S(n, n - 1)
***** Triangle case # 13 = 8,5,1,2 *****
Dim n = 8, A = 5, (u, v) = (1, 2), type = Stirling

##### Triangle array for dimension 9 #####

```
1
0  1
0  3  1
0  27 12  1
0  729 351 39 1
0  59049 29160 3510 120 1
0  14348907 7144929 882090 32670 363 1
0  10460353203 5223002148 650188539 24698520 297297 1092 1
0  22876792454961 11433166050879 1427185336941 54665851779 674887059 2685501 3279 1
```

%%%%% Diagonal differences for S(n, n-1) %%%%%

##### Triangle array for dimension 7 #####

```
3
12  9
39 27  18
120 81  54  36
363 243 162 108  72
1092 729 486 324 216  144
3279 2187 1458 972 648 432  288
```

Total readline of APnot_Cases.txt = 13

***** Triangle case # 1 = 6,1,2,1,3,6,10,15,21 *****
Dim n = 6, (u, v) = (1, 2), ips = [1, 3, 6, 10, 15, 21], type = Stirling

##### Triangle array for dimension 7 #####

```
1
0  1
0  1  1
0  3  4  1
0  18  27 10  1
0  180 288 127 20  1
0  2700 4500 2193 427 35 1
```

%%%%% Diagonal differences for S(n, n-1) %%%%%
##### Triangle array for dimension 5   #####

1  
4 3  
10 6 3  
20 10 4 1  
35 15 5 1 0  

***** Triangle case # 2 = 5,1,2,-3,-2,-1,0,1,2,3   *****  
Dim n = 5, (u, v) = (1, 2), ips = [-3, -2, -1, 0, 1, 2, 3], type = Stirling  

##### Triangle array for dimension 6   #####

1  
0 1  
0 -3 1  
0 6 -5 1  
0 -6 11 -6 1  
0 0 -6 11 -6 1  

%%%%% Diagonal differences for S(n, n-1)   %%%%%

##### Triangle array for dimension 4   #####

-3  
-5 -2  
-6 -1 1  
-6 0 1 0  

##### Triangle array for dimension 7   #####

1  
0 1  
0 2 1  
0 4 7 1  
0 8 39 15 1  
0 16 203 159 26 1  
0 32 1031 1475 445 40 1  

%%%%% Diagonal differences for S(n, n-1)   %%%%%

##### Triangle array for dimension 5   #####

2  
7 5
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**** Triangle case # 4 = 7,1,3,1,2,4,8,16,32,64,128,256 ****

Dim n = 7, (u, v) = (1, 3), ips = [1, 2, 4, 8, 16, 32, 64, 128, 256], type = Stirling

##### Triangle array for dimension 8 #####

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%%%%% Diagonal differences for S(n, n-1) %%%%%

##### Triangle array for dimension 6 #####

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**** Triangle case # 5 = 5,1,4,-6,-4,-2,0,2,4,6,8,10,12 ****

Dim n = 5, (u, v) = (1, 4), ips = [-6, -4, -2, 0, 2, 4, 6, 8, 10, 12], type = Stirling

##### Triangle array for dimension 6 #####

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%%%%% Diagonal differences for S(n, n-1) %%%%%

##### Triangle array for dimension 4 #####

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</table>
-24   -4   4
-24   0   4   0

***** Triangle case # 6 = 7,1,4,12,20,30,42,56,72,90,110,132,156,182,210 *****

Dim n = 7, (u, v) = (1, 4), ips = [2, 6, 12, 20, 30, 42, 56, 72, 90, 110, 132, 156, 182, 210], type = Stirling

#### Triangle array for dimension 8 ####
1
0   1
0   4   1
0   40   22   1
0   720   656   62   1
0   20160   26960   4004   132   1
0   806400   1476000   307240   15620   240   1
0   43545600   104126400   28513120   1994200   46820   394   1

%%%% Diagonal differences for S(n, n-1) %%%%%

#### Triangle array for dimension 6 ####
4
22   18
62   40   22
132   70   30   8
240   108   38   8   0
394   154   46   8   0   0

Total readline of InputSeq_Cases.txt = 6

Process finished with exit code 0
1. Dr. Lawrence K. Wang (王抗曝)

Lawrence K. Wang has over 30+ years of professional experience in facility design, environmental sustainability, natural resources, STEAM education, global pollution control, construction, plant operation, and management. He has expertise in water supply, air pollution control, solid waste disposal, drinking water treatment, waste treatment, and hazardous waste management.

He was the Director/Acting President of the Lenox Institute of Water Technology, Engineering Director of Krofta Engineering Corporation and Zorex Corporation, and a Professor of RPI/SIT/UIUC, in the USA.

He was also a Senior Advisor of the United Nations Industrial and Development Organization (UNIDO) in Austria.

Dr. Wang is the author of over 700 technical papers and 45+ books, and is credited with 24 US patents and 5 foreign patents.
He earned his two HS diplomas from the High School of National Taiwan Normal University and the State University of New York. He also earned his BS degree from National Cheng-Kung University, Taiwan, ROC, his two MS degrees from the University of Missouri and the University of Rhode Island, USA, and his PhD degree from Rutgers University, USA.

Currently he is the Chief Series Editor of the Handbook of Environmental Engineering series (Springer); Chief Series Editor of the Advances in Industrial and Hazardous Wastes Treatment series, (CRC Press, Taylor & Francis); co-author of the Water and Wastewater Engineering series (John Wiley & Sons); and Co-Series Editor of the Handbook of Environment and Waste Management series (World Scientific). Dr. Wang is active in professional activities of AWWA, WEF, NEWWA, NEWEA, AIChE, ACS, OCEESA, etc.

2. Dr. Hung-ping Tsao (曹恆平)

Hung-ping Tsao has been a mathematician, a university professor, and an assistant actuary, serving private firms and universities in the United States and Taiwan for 30+ years. He used to be an Associate Member of the Society of Actuaries and a Member of the American Mathematical Society. His research have been in the areas of college mathematics, actuarial mathematics, management mathematics, classic number theory and Sudoku puzzle solving.
In particular, bikini and open top problems are presented to share some intuitive insights and some type of optimization problems can be solved more efficiently and categorically by using the idea of the boundary being the marginal change of a well-rounded region with respect to its inradius; theory of interest, life contingency functions and pension funding are presented in more simplified and generalized fashions; the new way of the simplex method using cross-multiplication substantially simplified the process of finding the solutions of optimization problems; the generalization of triangular arrays of numbers from the natural sequence based to arithmetically progressive sequences based opens up the dimension of explorations; the introduction of step-by-step attempts to solve Sudoku puzzles makes everybody’s life so much easier and other STEAM project development.

Dr. Tsao is the author of 3 books and over 30 academic publications. Among all of the above accomplishments, he is most proud of solving manually in the total of ten hours the hardest Sudoku posted online by Arto Inkala in early July of 2012.

He earned his high school diploma from the High School of National Taiwan Normal University, his BS and MS degrees from National Taiwan Normal University, Taipei, Taiwan, his second MS degree from the UWM in USA, and a PhD degree from the University of Illinois, USA.
Editors of the eBook Series of the "EVLUTIONARY PROGRESS IN SCIENCE, TECHNOLOGY, ENGINEERING, ARTS AND MATHEMATICS (STEAM)"

Dr. Lawrence K. Wang (王抗曝) - left

Dr. Hung-ping Tsao (曹恆平) - right
Introduction to the E-BOOK Series of the "EVOLUTIONARY PROGRESS IN SCIENCE, TECHNOLOGY, ENGINEERING, ARTS AND MATHEMATICS (STEAM)" and This Chapter “EVOLUTIONARY MATHEMATICS AND SCIENCE FOR GENERAL FAMOUS NUMBERS: STIRLING-EULER-LAH-BELL”

The acronym STEM stands for “science, technology, engineering and mathematics”. In accordance with the National Science Teachers Association (NSTA), “A common definition of STEM education is an interdisciplinary approach to learning where rigorous academic concepts are coupled with real-world lessons as students apply science, technology, engineering, and mathematics in contexts that make connections between school, community, work, and the global enterprise enabling the development of STEM literacy and with it the ability to compete in the new economy”. The problem of this country has been pointed out by the US Department of Education that “All young people should be prepared to think deeply and to think well so that they have the chance to become the innovators, educators, researchers, and leaders who can solve the most pressing challenges facing our nation and our world, both today and tomorrow. But, right now, not enough of our youth have access to quality STEM learning opportunities and too few students see these disciplines as springboards for their careers.” STEM learning and applications are very popular topics at present, and STEM related careers are in great demand. According to the US Department of Education reports that the number of STEM jobs in the United States will grow by 14% from 2010 to 2020, which is much faster than the national average of 5-8 % across all job sectors. Computer programming and IT jobs top the list of the hardest to fill jobs.
Despite this, the most popular college majors are business, law, etc., not STEM related. For this reason, the US government has just extended a provision allowing foreign students that are earning degrees in STEM fields a seven month visa extension, now allowing them to stay for up to three years of “on the job training”. So, at present STEM is a legal term. The acronym STEAM stands for “science, technology, engineering, arts and mathematics”. As one can see, STEAM (adds “arts”) is simply a variation of STEM. The word of “arts” means application, creation, ingenuity, and integration, for enhancing STEM inside, or exploring of STEM outside. It may also mean that the word of “arts” connects all of the humanities through an idea that a person is looking for a solution to a very specific problem which comes out of the original inquiry process. STEAM is an academic term in the field of education.

The University of San Diego and Concordia University offer a college degree with a STEAM focus. Basically STEAM is a framework for teaching or R&D, which is customizable and functional, thence the “fun” in functional. As a typical example, if STEM represents a normal cell phone communication tower looking like a steel truss or concrete column, STEAM will be an artificial green tree with all devices hided, but still with all cell phone communication functions. This e-book series presents the recent evolutionary progress in STEAM with many innovative chapters contributed by academic and professional experts.

This e-book chapter, “EVOLUTIONARY MATHEMATICS AND SCIENCE FOR GENERAL FAMOUS NUMBERS: STIRLING-EULER-LAH-BELL” is Dr. Hung-ping Tsao’s collection of thoughts, works and articles about various ways of coming up with formulas for sums of powers throughout his retired period for seventeen years now. Three years prior to the publication of “EXPLICIT POLYNOMIAL EXPRESSIONS FOR SUMS OF POWERS OF AN ARITHMETIC PROGRESSION”, he gave a few talks among universities in Taiwan and a class of gifted students of his Alma Mater (High School of National Taiwan Normal University). He was then invited to present “General Triangular Arrays of Numbers” by “22nd Asian Technology Conference in Mathematics” (Chung Yuan Christian University, December 19, 2017). He is also grateful that Professor Ronald Graham [author of “CONCRETE MATHEMATICS”] replied promptly to my e-mails with two separate attachments of his manuscripts that he generalized most of the special functions in Chapter 6 of “CONCRETE MATHEMATICS”. He is presenting here a systemic but rather long account of his personal excursion into the realm of numbers initiated by Blaise Pascal, James Stirling, Leonhard Euler and Jacob Bernoulli, which is therefore not meant to be a categorical survey of the topic. Dr. Leon Chang joined in 2020 with his Python program to help verifying formulas and producing tables for further investigation.