Evolutionary mathematics and science for life contingency investigation

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This chapter will start with the accumulation function and use the geometric point of view to generalize and simplify the theory of interest. Then survey the laws of mortality from both points of view of stochastic theory and traditional actuaries. There is a thorough discussion and simple visualization of Balducci and uniform distribution fitting the Male Table of 1958 CSO are presented. Various complicated exposure formulas for a mortality study are obtained by a simple inspection of the valuation schedule in demography. Life insurance and annuities are first introduced in three different points of view: deterministic, stochastic and dynamic. Then the uniform representation of a general life contingency function and its derivative is defined in such a fashion that deferred, term, endowment, life insurance and life annuity with level or varying benefit and premium can be treated all in one shot.

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14. ABSTRACT

This chapter will start with the accumulation function and use the geometric point of view to generalize and simplify the theory of interest. Then survey the laws of mortality from both points of view of stochastic theory and traditional actuaries. There is a thorough discussion and simple visualization of Balducci and uniform distribution fitting the Male Table of 1958 CSO are presented. Various complicated exposure formulas for a mortality study are obtained by a simple inspection of the valuation schedule in demography. Life insurance and annuities are first introduced in three different points of view: deterministic, stochastic and dynamic. Then the uniform representation of a general life contingency function and its derivative is defined in such a fashion that deferred, term, endowment, life insurance and life annuity with level or varying benefit and premium can be treated all in one shot.
EVOLUTIONARY MATHEMATICS AND SCIENCE FOR
LIFE CONTINGENCY INVESTIGATION

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Hung-ping Tsao, PhD

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EVOLUTIONARY MATHEMATICS AND SCIENCE FOR
LIFE CONTINGENCY INVESTIGATION

Authored by: Hung-ping Tsao (曹恆平)

ABSTRACT

We shall start with the accumulation function and use the geometric point of view to generalize and simplify the theory of interest. Then survey the laws of mortality from both points of view of stochastic theory and traditional actuaries. There is a thorough discussion and simple visualization of Balducci and uniform distribution of deaths assumptions of mortality rates of fractional ages. Two least square-fit cubic survivorship functions for fitting the Male Table of 1958 CSO are presented. Various complicated exposure formulas for a mortality study are obtained by a simple inspection of the valuation schedule in demography. Life insurance and annuities are first introduced in three different points of view: deterministic, stochastic and dynamic. Then the uniform representation of a general life contingency function and its derivative is defined in such a fashion that deferred, term, endowment, life insurance and life annuity with level or varying benefit and premium can be treated all in one shot.

Keywords: Accumulation function, Balducci, Uniform distribution of deaths, Mortality rate, Survivorship function, Exposure formula, Valuation schedule, Life insurance, Life annuity, Life contingency function, Derivative, Deferred insurance, Term insurance, Endowment, Benefit, Premium, Life actuarial model, Compound interest, Expense factor, Premium, Reserve, Cash value, Present value, Future value, Maturity benefit, Age-at-death, Terminal age, Distribution function, Survival function, Risk, Probability density function, Migration, Discount function, Utility function, Derivative of life contingency function.
NOMENCLATURE

$\delta$ the force of interest

$\mu_x$ the force of mortality

$a(x)$ the accumulation function

$i$ the nominal rate of interest

$i^{(m)}$ the nominal rate of interest payable $m$ times a year

$d^{(m)}$ the nominal rate of discount payable $m$ times a year

$\dd{a}{n}{i}{(m)}$ the present value of an annuity due which pays $m^{-1}$ at the beginning of each $m$th of a year for $n$ years

$\dd{a}{n}{(m)}$ the present value of an annuity immediate which pays $m^{-1}$ at the end of each $m$th of a year for $n$ years

$\dd{a}{n}$ the present value of a continuous annuity payable continuously for $n$ years, with the total of 1 paid during each year

$\dd{s}{n}{(m)}$ the future value of an annuity due which pays $m^{-1}$ at the beginning of each $m$th of a year for $n$ years

$\dd{s}{n}{(m)}$ the future value of an annuity immediate which pays $m^{-1}$ at the end of each $m$th of a year for $n$ years
$\bar{s}_n$ the future value of a continuous annuity payable continuously for n years, with the total of 1 paid during each year

$X$ the random variable of a newborn’s age-at-death

$\omega$ the terminal age

$F(x)$ the distribution function (d. f.) of $X$

$S(x)$ the survival function

$(x)$ the life aged $x$

$\mu q_x$ the probability that a life $(x)$ aged $x$ will die between ages $x + t$ and $x + t + u$

$\mu q_x$ the probability that $(x)$ will die within $t$ years

$\mu p_x$ the probability that $(x)$ will survive for $t$ years

$q_x$ the probability that $(x)$ will die within a year

$p_x$ the probability that $(x)$ will survive for a year

$L(0)$ the cohort of newborns

$L(0)$ the number of newborns in $L(0)$

$L(x)$ those in $L(0)$ who survive to age $x$
\(l_x\) the number of lives in \(L(x)\)

\(n d_x\) the number of those in \(L(x)\) who will die within \(n\) years

\(d_x\) those in \(L(x)\) who will die within a year

\(P^z_x\) the number of persons aged between \(x\) and \(x + 1\) at the beginning of the calendar year \(z\)

\(E^z_x\) the number of persons attained age \(x\) during the calendar year \(z\)

\(\alpha D^z_x\) the number of deaths among \(E^z_x\) during the calendar year \(z\)

\(\alpha D^z_x\) the number of deaths among \(P^z_x\) before the attainment of age \(x + 1\)

\(\alpha m^z_x\) the number of migrants in addition to \(E^z_x\) during the calendar year \(z\)

\(\alpha m^z_x\) the number of migrants in addition to \(P^z_x\) before the attainment of age \(x + 1\)

\(A^1_{x:n}\) n-year term insurance of 1 payable at the end of the year of death

\(A^1_{x:n}\) n-year pure endowment of 1 payable at the end of the \(n\)th year when \((x)\) lives

\(A_{x:n}\) n-year endowment insurance of 1 payable either at the end of the year of death or at the end of the \(n\)th year when \((x)\) survives

\(a_{x:n}\) n-year annuity of 1 payable at the end of each year while \((x)\) survives
\( \ddot{a}_{x,n} \) n-year annuity of 1 payable at the beginning of each year while \((x)\) survives

\( \alpha_{x,n}^1 \) n-year term life contingency function with the death benefit \( \alpha_{k} \) payable at the end of the year of death

\( \alpha_{x,n}^1 \) n-year pure endowment of \( \alpha_n \) at the date of maturity

\( A_{x,n}^1 \) n-year term insurance of 1 payable at the end of the year of death

\( A_{x,n}^1 \) n-year pure endowment of 1 payable at the end of the nth year when \((x)\) lives

\( A_{x:n} \) n-year endowment insurance of 1 payable either at the end of the year of death or at the end of the nth year when \((x)\) lives

\( d_\delta (x,t) \) the discount function of interest

\( d_\mu (x,t) \) the discount function of mortality

\( \overline{\alpha}_{x,n}^h \) an h-year deferred n-year continuous life contingency function

\( \overline{\alpha}_{x}^h \) an h-year deferred whole life continuous contingency function

\( \overline{\alpha}_{x,n} \) an n-year continuous life contingency function

\( \overline{\alpha}_x \) a whole life continuous contingency function

\( \overline{P}(\overline{\alpha}_{x,n}^h) \) the continuously paid net level premium of \( \overline{\alpha}_{x,n}^h \), with payments for r years
an n-year continuous contingency function providing the present value of the
depth benefit \((t+1)\overline{a}_t\) at time \(t\) and the maturity benefit \(n\overline{a}_n\).

by \((I\overline{a})^1_{x:n}\) an n-year continuous contingency function providing the present value of the
depth benefit \((t+1)\overline{a}_t\) at time \(t\) and the maturity benefit 0

\((D\overline{a})^1_{x:n}\) an n-year continuous contingency function providing the present value of the
depth benefit \((n-t)\overline{a}_t\) at time \(t\)

\((D^{(m)}\overline{a})^1_{x:n}\) an mthly decreasing life contingency function

\((D^{(m)}_h\overline{a})^1_{x:n}\) an mthly decreasing life contingency function with the death benefit decreases
only for \(h\) years

1. INTRODUCTION

Some forty years ago, I started out my eight years of actuarial career as an actuarial student and
was provided by all my employers with study time and materials to prepare for Actuarial
Exams sponsored by SOA. By studying the only one textbook (1), which was teasingly called
“The Bible”, inside out, I personally invented many short cuts for solving various problems.
I later published thirteen papers (6)-(18), a lecture note (4) and a textbook (5) in Chinese,
which I did consult (2). Recently, I found out that my innovative ideas such as the uniform
representation of a general life contingency function and its derivative were not even
mentioned in (3). Therefore, I feel obliged to write this chapter for the benefit of readers.
A life actuarial model is based on three major factors: interest, mortality and expense. We first give a general way of constructing life actuarial models in terms of accumulation functions.

<table>
<thead>
<tr>
<th>Force of Interest $\delta$</th>
<th>Force of Mortality $\mu_s$</th>
<th>Expense Percentage $\varepsilon$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interest Related Accumulation Function $a_{\delta}(t) = e^{\delta t}$</td>
<td>Mortality Related Accumulation Function $a_{\mu_s}(t) = e^{\int_0^t \mu_s(\tau) d\tau}$</td>
<td></td>
</tr>
</tbody>
</table>

| Annuity $\overline{a}_{x:n} = \int_0^n [a_{\delta}(t)a_{\mu_s}(t)]^{-1} dt$ | Endowment Insurance $\overline{A}_{x:n} = \int_0^n [a_{\delta}(t)]^{-1} \frac{d}{dt} \{-[a_{\mu_s}(t)]^{-1}\} dt + [a_{\delta}(n)a_{\mu_s}(n)]^{-1}$ |

Net Level Premium $\overline{P}_{x:n} = \overline{A}_{x:n} / \overline{a}_{x:n}$

Gross Level Premium $\overline{G}_{x:n} = \overline{P}_{x:n} (1 + \varepsilon)$

Net Level Premium Reserve at Year $t$ $\overline{V}_{x:n} = \overline{A}_{x+t:n-t} - \overline{P}_{x:n} \overline{a}_{x+t:n-t}$

Cash Value at Year $t$ $\overline{C}_{x:n} = \overline{A}_{x+t:n-t} - \overline{P}_{x:n} \overline{a}_{x+t:n-t}$

Figure 1. The structure of an $n$ year continuous life actuarial model

In the model outlined in Figure 1, let

$$\overline{a}(t) = \int_0^n [a_{\delta}(u)]^{-1} du.$$ 

Then by using the integration by parts, we can write

$$\overline{a}_{x:n} = \int_0^n \overline{a}(t) \frac{d}{dt} \{-[a_{\mu_s}(t)]^{-1}\} dt + \overline{a}(n)[a_{\mu_s}(n)]^{-1},$$

so that all life contingency functions can be written as

$$\overline{a}_{x:n} = \int_0^n \overline{a}(t) \frac{d}{dt} \{-[a_{\mu_s}(t)]^{-1}\} dt + \overline{a}(n)[a_{\mu_s}(n)]^{-1},$$

where $\overline{a}(t)$ is the present value of the benefit at time $t$ so that the first term $\overline{a}_{x:n}^{-1}$ is the death benefit and the second term $\overline{a}_{x:n}^{-1}$ is the maturity benefit. This is generally true in any model.
2. THEORY OF COMPOUND INTEREST

2.1 Functions of compound interest

We shall start with the accumulation function and use the geometric point of view to generalize and simplify the theory of interest.

Let \( a(x) \) be an increasing positive function satisfying

\[ a(-x) = [a(x)]^{-1}. \quad \text{Eq. 1} \]

From Eq. 1, it follows that \( a(0) = 1 \) and that

\[ [a(0) - a(-x)]^{-1} - [a(x) - a(0)]^{-1} = 1, \quad x > 0. \quad \text{Eq. 2} \]

We shall use Eq. 2 to derive important formulas later. A continuous \( a(x) \) further requires the existence of \( a'(0) \) (denoted by \( \delta \)). For example,

\[ a(x) = (1+i)^x, \]

where \( i \) is the nominal rate of interest. We first list the notations and definitions of major functions with the illustrative Figures 2-7 as follows:

\( \text{i}^{(m)} = \) the nominal rate of interest payable \( m \) times a year

\( \text{d}^{(m)} = \) the nominal rate of discount payable \( m \) times a year

\( \delta = \) the force of interest

\( \ddot{a}_n^{(m)} = \) the present value of an annuity due which pays \( m^{-1} \) at the beginning of each \( m \)th of a year for \( n \) years

\[
\begin{array}{cccccccc}
& m^{-1} & m^{-1} & m^{-1} & \ldots & \ldots & \ldots & m^{-1} \\
0 & & & & & & & \\
\ddot{a}_n^{(m)} & & & & & & & \\
\end{array}
\]

\( n \)

Figure 2. The present value of \( \ddot{a}_n^{(m)} \)
\( a_n^{(m)} = \) the present value of an annuity immediate which pays \( m^{-1} \) at the end of each \( m \)th of a year for \( n \) years

\[
\begin{array}{cccccc}
m^{-1} & m^{-1} & m^{-1} & \ldots & \ldots & m^{-1} \\
0 & & & & & n \\
\end{array}
\]

\( a_n^{(m)} \)

Figure 3. The present value of \( a_n^{(m)} \)

\( \bar{a}_n = \) the present value of a continuous annuity payable continuously for \( n \) years, with the total of 1 paid during each year

\[
\begin{array}{cccccc}
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 1 & 2 & 3 & \ldots & n-1 & n \\
\bar{a}_n \\
\end{array}
\]

Figure 4. The present value of \( \bar{a}_n \)

\( \ddot{a}_n^{(m)} = \) the future value of \( \ddot{a}_n^{(m)} \)

\[
\begin{array}{cccccc}
m^{-1} & m^{-1} & m^{-1} & \ldots & \ldots & m^{-1} \\
0 & & & & & n \\
\end{array}
\]

\( \ddot{a}_n^{(m)} \)

Figure 5. The future value of \( \ddot{a}_n^{(m)} \)

\( \ddot{s}_n^{(m)} = \) the future value of \( \ddot{a}_n^{(m)} \)

\[
\begin{array}{cccccc}
m^{-1} & m^{-1} & m^{-1} & \ldots & \ldots & m^{-1} \\
0 & & & & & n \\
\end{array}
\]

\( \ddot{s}_n^{(m)} \)

Figure 6. The future value of \( \ddot{a}_n^{(m)} \)

\( s_n^{(m)} = \) the future value of \( a_n^{(m)} \)

\[
\begin{array}{cccccc}
m^{-1} & m^{-1} & m^{-1} & \ldots & \ldots & m^{-1} \\
0 & & & & & n \\
\end{array}
\]

\( s_n^{(m)} \)

Figure 6. The future value of \( a_n^{(m)} \)
\( \bar{s}_n = \) the future value of \( \bar{a}_n \)

\[
\begin{array}{cccccccc}
0 & 1 & 2 & 3 & \ldots & \ldots & n-1 & n \\
\bar{s}_n
\end{array}
\]

Figure 7. The present value of \( \bar{a}_n \)

When \( m = 1 \), the superscript \( (m) \) of the above notations is simply dropped.

On the other hand, we can generalize the definition of the force of interest at time \( x \) to be

\[
\delta(x) = \lim_{t \to 0} \left\{ \frac{[a(x+t) - a(x)]}{a(x)} \right\}/t .
\]

Then

\[
\delta(x) = a'(x)/a(x)
\]

and

\[
\delta = \delta(0).
\]

In the case that

\[
a(x) = (1+i)^x,
\]

we have

\[
\delta(x) = \lim_{t \to 0} \left\{ \frac{[a(x+t) - a(x)]}{a(x)} \right\}/t = \lim_{t \to 0} [(1+i)^t - 1]/t = \delta
\]

for all \( x \) and

\[
\delta = \ln (1+i)
\]

or

\[
i = e^\delta - 1.
\]
2.2 Geometric point of view

Let $a(x)$ be an accumulation function. Define $i^{(m)}$ to be the slope of the line joining $(0, 1)$ and $(m^{-1}, a(m^{-1}))$. Let $d^{(p)} = i^{(-p)}$. Then $d^{(p)}$ is the slope of the line joining $(0, 1)$ and $(-p^{-1}, a(-p^{-1}))$.

![Figure 8. Geometric visualization of the force of interest](image)

We can visualize from Figure 8 that

$$d^{(p)} < \delta < i^{(m)}$$

and

$$\lim_{p \to \infty} d^{(p)} = \delta = \lim_{m \to \infty} i^{(m)}.$$ 

The latter can also be derived as follows according to the definitions of $d^{(p)}$, $\delta$ and $i^{(m)}$:

$$\lim_{p \to \infty} d^{(p)} = \lim_{m \to \infty} \frac{[a(-p^{-1}) - a(0)]}{(-p^{-1})} = a'(0) = \delta = \lim_{m \to \infty} \frac{[a(m^{-1}) - a(0)]}{(m^{-1})} = \lim_{m \to \infty} i^{(m)}.$$ 

![Figure 9. Geometric visualization of annuity functions](image)
Referring to Figure 9, we define the following annuity functions:

\[ a^{(m)}_n = \text{the ratio of the length of } P_nQ_n \text{ to the slope of } OP^{-m}_m \]

\[ a^{(m)}_n = \text{the ratio of the length of } P_nQ_n \text{ to the slope of } OP^{-m}_m \]

\[ \overline{a}_n = \text{the ratio of the length of } P_nQ_n \text{ to the slope of the tangent at } O \]

\[ s^{(m)}_n = \text{the ratio of the length of } P_nQ_n \text{ to the slope of } OP^{-m}_m \]

\[ s^{(m)}_n = \text{the ratio of the length of } P_nQ_n \text{ to the slope of } OP^{-m}_m \]

\[ \overline{s}_n = \text{the ratio of the length of } P_nQ_n \text{ to the slope of the tangent at } O \]

With the exception of continuous functions, the above can also be defined as follows:

\[ a^{(m)}_n = \text{the ratio of the length of } P_nQ_n \text{ to } m \text{ times that of } P_{-m}Q_{-m} \]

\[ a^{(m)}_n = \text{the ratio of the length of } P_nQ_n \text{ to } m \text{ times that of } P_{-m}Q_{-m} \]

\[ \overline{s}^{(m)}_n = \text{the ratio of the length of } P_nQ_n \text{ to } m \text{ times that of } P_{-m}Q_{-m} \]

\[ s^{(m)}_n = \text{the ratio of the length of } P_nQ_n \text{ to } m \text{ times that of } P_{-m}Q_{-m} \]

Hence

\[ a^{(m)}_n = [1 - a(-nm)]/d^{(m)} = [1 - a(-nm)]/(m[1 - a(-m^{-1})]) \],

\[ a^{(m)}_n = [1 - a(-nm)]/i^{(m)} = [1 - a(-nm)]/(m[a(m^{-1}) - 1]) \],

\[ \overline{a}_n = [1 - a(-n)]/a'(0) = [1 - a(-n)]/\delta, \]

\[ s^{(m)}_n = [a(nm) - 1]/d^{(m)} = [a(nm) - 1]/(m[1 - a(-m^{-1})]) \],

\[ s^{(m)}_n = [a(nm) - 1]/i^{(m)} = [a(nm) - 1]/(m[a(m^{-1}) - 1]) \],

\[ \overline{s}_n = [a(n) - 1]/a'(0) = [a(n) - 1]/\delta. \]
In conclusion, we shall derive the following important formulas from Eqs. 1 and 2.

\[ i^{(m)} = a(m^{-1})d^{(m)}; \]  
Eq. 3

\[ [d^{(m)}]^{-1} - [i^{(m)}]^{-1} = m^{-1}; \]  
Eq. 4

\[ [\hat{a}^{(m)}]^{-1} - [\hat{s}^{(m)}]^{-1} = d^{(m)}; \]  
Eq. 5

\[ [a^{(m)}]^{-1} - [s^{(m)}]^{-1} = i^{(m)}; \]  
Eq. 6

\[ [\bar{a}^{(m)}]^{-1} - [\bar{s}^{(m)}]^{-1} = \delta. \]  
Eq. 7

Since \( a(0) = 1 \), we can derive Eq. 3 from Eq. 1 as follows.

\[ \frac{1}{d^{(m)}} - \frac{1}{i^{(m)}} = \frac{a(nm)}{a(0)} - \frac{a(0)}{a(-nm)} = \frac{1}{m}. \]

Eq. 4 can be derived from Eq. 3 as follows.

\[ \frac{1}{\hat{a}^{(m)}} - \frac{1}{\hat{s}^{(m)}} = d^{(m)} = \frac{a(m^{-1}) - a(0)}{a(0) - a(-nm)} = d^{(m)}. \]

From Eq. 2, we can derive Eq. 5 [similar for Eqs. 6 and 7] as follows.

\[ \frac{1}{\hat{a}^{(m)}} - \frac{1}{\hat{s}^{(m)}} = \frac{d^{(m)}}{a(0) - a(-nm)} - \frac{d^{(m)}}{a(nm) - a(0)} = d^{(m)}. \]

In the case that \( a(x) = (1+i)^x \), we have

\[ \hat{a}_n = [1 - (1+i)^{-n}] / d = \hat{s}_n (1+i)^{-n}; \]

\[ a_n = [1 - (1+i)^{-n}] / i = s_n (1+i)^{-n}; \]

\[ \bar{a}_n = [1 - (1+i)^{-n}] / \delta = \bar{s}_n (1+i)^{-n}. \]

From these formulas we can readily derive Eqs. 5-7 for the case that \( m = 1 \), which can also be visualized from Figures 10-12.
n-year payments of annuity due future value

$$
\frac{1}{s_n} \quad \frac{1}{s_n} \quad \frac{1}{s_n} \ldots \ldots \ldots \frac{1}{s_n} \quad 1
\]

$$

$$
+ \quad d \quad d \quad d \ldots \ldots \ldots \quad d \quad d \frac{s_n}{a_n} = \frac{1}{\ddot{a}_n} \quad \frac{1}{\ddot{a}_n} \quad \frac{1}{\ddot{a}_n} \ldots \ldots \frac{1}{\ddot{a}_n} \quad (1 + i)^n
\]

$$

$$
(1/\ddot{a}_n) \frac{s_n}{a_n} = (1 + i)^n = 1 + d \frac{s_n}{\ddot{s}_n} = (1/\ddot{s}_n) \frac{\ddot{s}_n}{s_n} + d \frac{s_n}{a_n}
\]

Figure 10. Future value of n-year payments of annuity due

n-year payments of annuity immediate future value

$$
\frac{1}{s_n} \quad \frac{1}{s_n} \quad \frac{1}{s_n} \ldots \ldots \ldots \frac{1}{s_n} \quad 1 + i
\]

$$

$$
+ \quad i \quad i \quad i \ldots \ldots \ldots \quad i \quad i \frac{s_n}{a_n} = \frac{1}{a_n} \quad \frac{1}{a_n} \quad \frac{1}{a_n} \ldots \ldots \frac{1}{a_n} \quad (1 + i)^{n+1}
\]

$$

$$
(1/a_n) \frac{s_n}{a_n} = (1 + i)^{n+1} = (1 + i) + [ (1 + i)^{n+1} - (1 + i)] = (1/s_n) \frac{s_n}{\ddot{s}_n} + i \frac{s_n}{\ddot{s}_n}
\]

Figure 11. Future value of n-year payments of annuity immediate

n-year payments of continuous annuity future value

$$
\frac{1}{s_n} \quad \frac{1}{s_n} \quad \frac{1}{s_n} \ldots \ldots \ldots \frac{1}{s_n} \quad \delta/d
\]

$$

$$
+ \quad \delta \quad \delta \quad \delta \ldots \ldots \ldots \quad \delta \quad \delta \frac{s_n}{a_n} = \frac{1}{\bar{a}_n} \quad \frac{1}{\bar{a}_n} \quad \frac{1}{\bar{a}_n} \ldots \ldots \frac{1}{\bar{a}_n} \quad (\delta/d)(1 + i)^n
\]

$$

$$
(1/\bar{a}_n) \frac{s_n}{a_n} = (\delta/d)(1 + i)^n = \delta/d + (\delta/d)[ (1 + i)^n - 1] = (1/\bar{s}_n) \frac{s_n}{\ddot{s}_n} + \delta \frac{s_n}{\ddot{s}_n}
\]

Figure 12. Future value of n-year payments of continuous annuity
3. LAWS OF MORTALITY

3.1 Point of view of stochastic theory

Let us first introduce the conventional notations as follows.

$X$: The random variable of a newborn’s age-at-death.

$\omega$: The terminal age.

$F(x)$: The distribution function (d. f.) of $X$.

$S(x) = 1 - F(x)$: The survival function.

Let $q_x$ be the probability that a life aged $x$ will die between ages $x + t$ and $x + t + u$. Then

$$q_x = \Pr[x + t < X \leq x + t + u | x < X \leq \omega] = \frac{F(x + t + u) - F(x + t)}{F(\omega) - F(x)} = \frac{S(x + t) - S(x + t + u)}{S(x) - S(\omega)},$$

where $F(\omega) = 1$ and $S(\omega) = 0$. The above relationships can be visualized from Figure 13.

![Figure 13](image_url)  
**Figure 13.** Linear visualization of the death rate $q_x$

When $u = 1$, we have (by omitting 1)

$$q_x = \Pr[x + t < X \leq x + t + 1 | x < X \leq \omega] = \frac{F(x + t + 1) - F(x + t)}{1 - F(x)} = \frac{S(x + t) - S(x + t + 1)}{S(x)}$$

and when $t = 0$, we have (by omitting 0 and replacing $u$ by $t$)

$$q_x = \Pr[x < X \leq x + t | x < X \leq \omega] = \frac{F(x + t) - F(x)}{1 - F(x)} = \frac{S(x) - S(x + t)}{S(x)}$$

Let $p_x = 1 - q_x$ be the probability that $(x)$ will survive for $t$ years. Then

$$q_x = \frac{q_{x+u} - q_x}{p_x} = iP_x - iP_xP_{x+u} = iP_x - iP_{x+u}P_{x+t} = iP_{x+u}q_{x+t}.$$
3.2 Point of view of traditional actuaries

In a life table, we can always find \( q_x \), \( x = 0, 1, 2, 3, \ldots \) \( \omega \), which is the probability that \( x \) will die within a year, namely \( q_x \) as introduced in the previous section; while \( p_x = 1 - q_x \) is the probability that \( x \) will survive in a year.

Let \( S(x) \) be a survival function. Then

\[
q_x = \frac{S(x) - S(x + 1)}{S(x)}
\]

and

\[
p_x = \frac{S(x + 1)}{S(x)}.
\]

Let \( L(0) \) be a cohort of \( l_0 \) newborns. Then the survivorship function \( l_x = l_0 S(x) \) is the number of those in \( L(0) \) who survive to age \( x \). Let \( L(x) \) denote such a set.

In this manner, the number of survivors can be tracked down as follows:

\[
l_1 = l_0 S(1) = l_0 p_0,
\]

\[
l_2 = l_0 S(2) = l_0 p_0 = l_0 p_0 p_1 = l_1 p_1,
\]

\[
l_3 = l_0 S(3) = l_0 p_0 p_1 = l_0 p_0 p_1 p_2 = l_2 p_2,
\]

\[
\vdots
\]

\[
l_\omega = l_0 S(\omega) = l_0 p_0 = l_0 p_0 p_1 p_2 \cdots p_{\omega-1} = l_\omega = l_0 p_{\omega-1} = 0.
\]

Let \( \pi d_x \) be the number of those in \( L(x) \) who will die within \( n \) years and let \( d_x \) be those in \( L(x) \) who will die within a year. Then
\[ n d_x = l_x - l_{x+n}, \]
\[ d_x = l_x - l_{x+1}, \]
\[ n q_x = n d_x / l_x, \]
\[ q_x = d_x / l_x, \]
\[ n p_x = (l_x + n d_x) / l_x, \]
\[ p_x = (l_x + d_x) / l_x, \]
\[ t+u q_x - q_x = \int_0^t q_x = \frac{d_{x+t}}{l_x} = \frac{(l_{x+t} / l_x) (d_{x+t} / l_{x+t})}{p_x u q_{x+t}}, \]

The above relationships can be visualized in Figure 14.

![Figure 14. Linear visualization of various death rates](#)

Now let us look at the instantaneous rate of mortality

\[ \mu_x = \lim_{t \to 0} (q_x / t), \]

Eq. 8

called the force of mortality. Since \[ q_x = \frac{S(x) - S(x+t)}{S(x)}, \]
we have

\[ \mu_x = \frac{S'(x)}{S(x)} = \frac{F'(x)}{S(x)} = \frac{F'(x)}{x p_0}. \]

Eq. 9

Hence \[ x p_0 \mu_x \] is the p. d. f. (probability density function) of \( X \).
4. MORTALITY RATES OF FRACTIONAL AGES

When $0 < t < 1$, $i_qx$ cannot be found in a life table. The following two methods are commonly used to solve this problem.

1) Linear interpolation:

$$l_{x+t} = (1-t)l_x + tl_{x+1}.$$ 

2) Reciprocal interpolation:

$$l_{x+t}^{-1} = (1-t)l_x^{-1} + tl_{x+1}^{-1}.$$ 

The first method assumes the uniform distribution of deaths throughout a year, called U-Assumption; while the second method is due to Balducci, called B-Assumption.

**Dual Theorem.** $l_x$ imposes B-Assumption if and only if $l_{x-t}$ imposes U-Assumption.

**Proof.** We shall assume that $l_{x-t}$ is close to 0 but not 0 to avoid $l_{x-t}^{-1}$ being undefined. Let

$$l_x^* = l_{x-t}^{-1}.$$ 

Then the theorem can be proved as follows:

$$l_{x+t}^* = l_{x-t}^{-1} = l_{x-t}^{-1} = [1 - (1-t)]l_{x-t}^{-1} + (1-t)l_{x-t}^{-1} = tl_{x+t}^* + (1-t)l_x^*.$$ 

Now, we shall derive the formula of $h q_{x+t}$ for the following two cases.

**Case 1.** $l_x$ imposes U-Assumption.

The following formula can also be visualized from Figure 15.

$$h q_{x+t} = \frac{l_{x+t} - l_{x+t+h}}{l_{x+t}} = \frac{(1-t)l_x + tl_{x+1} - (1-t-h)l_x - (t+h)l_{x+1}}{(1-t)l_x + tl_{x+1}} = \frac{[h(l_x - l_{x+1})]/l_x}{[l_x - t(l_x - l_{x+1})]/l_x} = \frac{h q_x}{1-tq_x}.$$
It follows from

\[ t \ p_x \ h \ q_{x+t} = h \ q_x \]

that

\[ h \ q_{x+t} = \frac{h \ q_x}{1-t \ q_x} = \frac{h \ q_x}{1-tq_x} \]  \hspace{1cm} \text{Eq. 10} \]

and that

\[ \mu_{x+t} = \lim \left( \frac{h \ q_{x+t}}{h} \right) = \frac{q_x}{(1-tq_x)}. \]

By taking \( t = 0 \) in Eq. 10, we have

\[ h \ q_x = h \ q_x \]

and by taking \( h = 1-t \), we have

\[ (1-t) q_{x+t} = \frac{(1-t) q_x}{1-tq_x}. \]

**Case 2.** \( l_x \) imposes B-Assumption.

According to Dual Theorem, \( l_x^* \) imposes U-Assumption. From Case 1, we have

\[ h \ q^*_{[\omega-(x+1)+(1-t-h)]} = \frac{h \ q^*_{\omega-(x+1)}}{1-(1-t-h)q^*_{\omega-(x+1)}}. \]
Since
\[
q_{x}^{*} = \frac{l_{x}^{*} - l_{x-1}^{*}}{l_{x+1}^{*} - l_{x}^{*}} = \frac{l_{x+1} - l_{x}}{l_{x+1}} = \frac{l_{x} - l_{x+1}}{l_{x}} = q_{x},
\]
it follows that
\[
hq_{x+t} = \frac{l_{x+t} - l_{x+t+h}}{l_{x+t}} = \frac{l_{x+t+h} - l_{x+t}}{l_{x+t}} = \frac{l_{x}^{*} - l_{x}^{*}}{l_{x+1}^{*}} = hq_{x}^{*} = hq_{[x+1]}^{*} = hq_{[x+1]h}^{*} = hq_{x}^{*} = \frac{hq_{x}}{1-(1-t-h)q_{x}}.
\]
Eq. 11 can be visualized from Figure 15 and Figure 16.

Figure 16. Linear visualization of the death rate in Case 2

To simplify the matter, we can further combine Figure 15 and Figure 16 into Figure 17 as though the time is running from \(x+1\) to \(x\) (having in mind that the time is actually running from \(x+1\) to \(x\)).

\[
q_{x}^{*} = \frac{l_{x}^{*} - l_{x-1}^{*}}{l_{x+1}^{*} - l_{x}^{*}} = \frac{l_{x+1} - l_{x}}{l_{x+1}} = \frac{l_{x} - l_{x+1}}{l_{x}} = q_{x},
\]

By taking \(t = 0\) in Eq. 11, we have
and by taking $h = 1 - t$, we have

$$1_q = (1-t)q_x.$$  \hspace{1cm} \text{Eq. 12}$$

Similar to Case 1, we can also derive

$$\mu_{x+t} = \lim_{h \to 0} \left( \frac{1}{h} \frac{q_{x+t}}{q_x} \right) = q_x / [1 - (1-t)q_x].$$

Because of the simplicity of Eq. 12, B-Assumption is often used as the basis of calculating the mortality rates.

5. MODELS OF THE SURVIVALSHIP FUNCTION

Mathematicians have long been searching for appropriate models for the survivorship function $l_x$. From Eq. 9, we can derive

$$l_x = l_0 S(x) = l_0 \frac{S(x)}{S(0)} = l_0 e^{\frac{\int_{S(x)} \ln S(y)}{S(0)}} = l_0 e^{\int_{y}^{x} \frac{S(y)}{S(y)} dy} = l_0 e^{-\int_{y}^{x} \mu_{y} dy}. \hspace{1cm} \text{Eq. 13}$$

In 1724, de Moivre first introduced

$$\mu_x = \frac{1}{x}$$

as the basis of the model, the survivorship function of which can be calculated from Eq. 13 as

$$l_x = l_0 \left( 1 - \frac{x}{\omega} \right),$$

where $\omega = 105$. This was used in those days for simplifying the calculation of life annuities primarily only for the range of ages from 12 to 86. This model was later generalized to

$$\mu_x = \frac{\alpha}{\omega - x}, \quad \alpha > 0,$$
the survivorship function of which being

\[ l_x = l_0 (1 - \frac{x}{\omega})^\alpha. \]

In 1825, Gompertz believed that the force of mortality was increasing in geometric progression and introduced

\[ \mu_x = Bc^x, \]  

Eq. 14

the survivorship function of which being

\[ l_x = l_0 e^{\frac{-B}{\ln(c^\prime - 1)}}. \]

By suitably adjusting \( B \) and \( c \), this model could fit the range of ages from 10 to 55. Therefore, it was used to construct the 1937 Standard Annuity Table.

In 1860, Makeham further generalized the model to

\[ \mu_x = A + Bc^x, \]  

Eq. 15

and later to

\[ \mu_x = A + Hx + Bc^x, \]

the survivorship function of which being

\[ l_x = l_0 e^{\frac{-B}{\ln(c^\prime - 1)} - Ax - \frac{Hx^2}{2} \frac{Bc^x}{\ln c}}. \]

By suitably adjusting \( A, B \) and \( c \), this model could fit any age over 20 and was used to construct the Commissioners 1941 Standard Ordinary Mortality Table and also the Annuity Table for 1949. Furthermore, both Eqs. 14 and 15 are often used nowadays to simplify compound probability problems involving multiple life insurance.
Later, the model based on
\[ \mu_x = \frac{Ac^x}{1 + Bc^x}, \]
the survivorship function of which being
\[ l_x = l_0 \left( \frac{1 + B}{1 + Bc^x} \right)^{\frac{A}{\ln c}}. \]

In 1939, Weibull introduced
\[ \mu_x = kx^\alpha \]
the survivorship function of which being
\[ l_x = l_0 e^{\frac{k}{\alpha+1}x^{\alpha+1}}. \]

In 1997, the author obtained the following two least-square-fit cubic survivorship functions:
\[ l_x^O = l_0 \left[ -\frac{1}{365} \left( \frac{x-46}{10} \right)^3 - \frac{1}{45} \left( \frac{x-46}{10} \right)^2 - \frac{1}{15} \left( \frac{x-46}{10} \right) + \frac{1}{1.115} \right]; \quad \text{Eq. 16} \]
\[ l_x^E = l_0 \left[ -\frac{1}{1970} \left( \frac{x-41}{5} \right)^3 - \frac{1}{205} \left( \frac{x-41}{5} \right)^2 - \frac{1}{55} \left( \frac{x-41}{5} \right) + \frac{1}{1.075} \right]. \quad \text{Eq. 17} \]

The function Eq. 16 was derived based on \( l_6, l_{16}, l_{26}, \ldots l_{86} \) of 1958 CSO Male Life Table and fit well the range of ages from 0 to 70.

The function Eq. 17 was derived based on \( l_6, l_{16}, l_{26}, \ldots l_{76} \) of the same table and fit well the range of ages from 3 to 79.

The error for each of these models is within 1% for most of the ages described above and about 2% for few ages as can be seen in the following comparison chart (Tables 1 and 2 combined).
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Table 2. Second half of the comparison chart
6. SIMPLE VISUALIZATIONS FOR SCHEDULE EXPOSURE FORMULAS

We shall introduce the method of valuation schedule in demography to be used to calculate the mortality rates for any observed group in the insurance industry such as the insured of a life insurance company, the annuitants of an annuity contract or the participants of a pension plan.

To undertake a mortality study for such a group, we need to specify the observation period and the mechanism of calculating the exposure and deaths. These calculations involve with starters, new entrants, withdrawers, deaths and enders.

For a large group, the valuation schedule exposure formulas are often considered rather than the individual record exposure formulas because of the obvious reason. These formulas are based only on the observed deaths and the periodic numeration of the individuals in the observed group, which are readily available from the data for the valuation purpose just as in the population study in demography.

We first adopt pertinent notations from the demography.

\[ P^z_x = \text{the number of persons aged between } x \text{ and } x + 1 \text{ at the beginning of the calendar year } z; \]
\[ E^z_x = \text{the number of persons attained age } x \text{ during the calendar year } z; \]
\[ \sigma D^z_x = \text{the number of deaths among } E^z_x \text{ during the calendar year } z; \]
\[ \sigma D^z_x = \text{the number of deaths among } P^z_x \text{ before the attainment of age } x + 1; \]
\[ D^z_x = \text{the number of deaths at age } x \text{ last birthday during the calendar year } z; \]
\[ D^z_{x\mid i} = \sigma D^z_{x-1} + \alpha D^z_x; \]
\[ D^z_{x\mid i+1} = \alpha D^z_x + \sigma D^z_{x+1}; \]
\[ a \, m^x_z = \text{the number of migrants in addition to } E^x_z \text{ during the calendar year } z; \]

\[ \delta \, m^x_z = \text{the number of migrants in addition to } P^x_z \text{ before the attainment of age } x + 1. \]

Figure 18. The observed deaths and the periodic numeration of the individuals

From the above figure, we have

\[ a \, m^x_z = E^x_z - P^{x+1}_z + a \, D^x_z \quad \text{Eq. 18} \]

and

\[ \delta \, m^x_z = P^x_z - E^{x+1}_z + \delta \, D^x_z. \quad \text{Eq. 19} \]
The number of migrants is the number of new entrants minus the number of withdrawers. In the insurance industry, the migration can be assumed to occur either at the insured's birthday or at the end of calendar year. Different migration and mortality assumptions will lead to different exposure formulas. The mortality rate is calculated as the ratio of the number of deaths over the total exposure. The treatment of deaths plays the major role in the calculation of different exposure formulas as discussed below. Let k be the number of months after January 1 for the average birthday of an observed group. For a large group, k is usually assumed to be 6. If the observation period is the calendar year z, we can group the deaths by age last birthday or by calendar age. If the observation period is from birthday in z to the birthday in z + 1, then the grouping of deaths is always by age last birthday.

Case 1. Calendar year study, deaths by age last birthday.

In the following figure, we assume that $a m^z_x$ occurs m months after January 1 and $d m^z_x$ occurs n months after January 1.

![Figure 19. Visualization of Case 1](image-url)
1) B-Assumption on deaths

Using the idea of potential and cancelled exposure, we can obtain

\[ E^x q_x + a_m \frac{m_z}{12} - \frac{m_{x+1}}{12} q_{x+1} + P_x^{z+1} q_{x+1} + P_x^{z-k} q_{x-k} + \delta m_{x+11} = D_x^z, \]

It follows from Eqs. 12, 18 and 19 that

\[ [E_x^z + (P_x^{z+1} + \alpha D_x^z - E_x^z) \frac{12-n}{12}] q_x = D_x^z. \]

Hence

\[ q_x = \frac{D_x^z}{E}, \]

where

\[ E = \frac{m}{12} E_x^z + \frac{n-k}{12} P_x^z + \frac{12-n}{12} E_x^{z+1} + \frac{k-m}{12} P_x^{z+1} + \frac{12-m}{12} D_x^z + \frac{12-n}{12} \delta D_x^z. \]

If the migration occurs on birthdays (m = 0 and n = 12), then

\[ E = \frac{12-k}{12} P_x^z + \frac{k}{12} P_x^{z+1} + \frac{k}{12} \delta D_x^z, \]

which can be visualized directly from the diagram below.

![Diagram](image)

Figure 20. The migration occurs on birthdays under B-Assumption
The coefficient (exposure) of $P_x^z$ is the length of the line segment $X$---$O$, the coefficient of $P_x^{z+1}$ is the length of the line segment $O$---$X$ and the coefficient of $\alpha D_x^z$ is the length of the line segment $O$---$O$.

If the migration occurs at year-ends ($m = n = k$), then

$$E = \frac{k}{12} E_x^z + \frac{12-k}{12} E_{x+1}^z + \frac{12-k}{12} D_x^z,$$

which can be visualized directly from the diagram below.

![Diagram](image)

Figure 21. The migration occurs at year-ends under B-Assumption

The coefficient of $E_{x+1}^z$ is the length of the line segment $X$---$O$, the coefficient of $E_x^z$ is the length of the line segment $O$---$X$, the coefficient of $\alpha D_x^z$ is the length of the line segment $X$---$O$ and the coefficient of $\alpha D_x^z$ is the length of the line segment $X$---$O$.

2) U-Assumption on deaths

Since the equivalent formula to Eq. 10 is far more complicated under U-Assumption, we shall use the direct approach by tracing down the deaths from segment to segment in the original diagram to obtain
\begin{equation}
E_x^{z+1} q_x + \alpha m_x^{z+\alpha} _{x-m} + P_x^{z+1} q_{x+k} + \beta m_x^{z+\beta} _{x+n} = D_x^z.
\end{equation}

Eq. 20

Due to the fact that U-Assumption is usually accompanied with the migration assumption either on birthdays or at year-ends, we shall only discuss these two special cases. If the migration occurs on birthdays (m = 0 and n = 12), then

\[(E_x^{z+1} \alpha m_x^z) \frac{k}{12} p_x = P_x^{z+1}.
\]

We shall make use of the following identity

\[k \frac{p_x}{12} q_{x-\frac{k}{12}} = k \frac{p_x}{12} (1 - \frac{k}{12} p_x) = k \frac{p_x}{12} - q_x = q_x - q_x.
\]

Eq. 21

By multiplying \( \frac{k}{12} p_x \) to Eq. 20 and making use of the above, we can obtain

\[P_x^{z+1} q_x + P_x^{z} (q_x - q_x) + \beta m_x^{z+\beta} _{x} p_x \frac{q_x+1}{12} = D_x^z (1 - q_x).
\]

It then follows from U-Assumption, as can be visualized from the figure below, that

\[E = \frac{12-k}{12} P_x^z + \frac{k}{12} P_x^{z+1} + \frac{k}{12} D_x^z,
\]

Figure 22. The migration occurs at year-ends under U-Assumption
The coefficient of $P^z_x$ is the length of the line segment O---X, the coefficient of $P^{z+1}_x$ is the length of the line segment X---O, the coefficient of $\alpha D^z_x$ is the length of the line segment X---O and the coefficient of $\delta D^z_x$ is the length of the line segment X---O. If the migration occurs at year-ends ($m = n = k$), by multiplying $\frac{k}{12} p_x$ to Eq. 20 and making use of Eq. 19, we can obtain

$$
(E^z_x \frac{k}{12} p_x k q_x + \frac{a m^z_x}{12} p_{x+1} q_x + (P^z_x + \delta m^z_x - \delta D^z_x)(q_{x_k} - q_{x_k}) + \delta D^z_x q_x = D^z_x.
$$

Since $E^z_x \frac{k}{12} p_x = P^{z+1}_x$, by applying Eq. 19 and U-Assumption to the above we can obtain

$$
E = \frac{k}{12} E^z_x + \frac{12 - k}{12} E^{z+1}_x + \delta D^z_x,
$$

which can be visualized directly from the figure below.

**Figure 23.** The migration occurs at year-ends under U-Assumption
The coefficient of $E_{x+1}^z$ is the length of the line segment O---X, the coefficient of $E_x^z$ is the length of the line segment X---O, and the coefficient of $aD_x^z$ is the length of the line segment O---O. The derivation of exposure formulas for the last two of the following cases is similar to the first and therefore will be omitted. However, we shall summarize the formulas of the case with accompanying figures and follow suit.

**Case 1.** Calendar year study, deaths by age last birthday.

**Case 2.** Calendar year study, deaths by calendar year.

**Case 3.** Birthday to birthday study, deaths by age last birthday.

**Case 1.** Calendar year study, deaths by age last birthday.

**Exposure formulas**

**B-Assumption on deaths**

**U-Assumption on deaths**

\[
E = \frac{12 - k}{12} P_x^z + \frac{k}{12} P_{x+1}^z + \frac{k}{12} aD_x^z
\]

\[
E = \frac{12 - k}{12} P_x^z + \frac{k}{12} P_{x+1}^z + \frac{k}{12} aD_x^z
\]

Figure 24. Calendar year study, deaths by age last birthday, migration on birthdays
Case 2. Calendar year study, deaths by calendar year.

Exposure formulas

**B-Assumption on deaths**

$$E = \frac{k}{12} E^z_x + \frac{12-k}{12} E^z_{x+1} + \frac{12-k}{12} D^z_x$$

**U-Assumption on deaths**

$$E = \frac{k}{12} E^z_x + \frac{12-k}{12} E^z_{x+1} + \frac{12-k}{12} D^z_x$$

Figure 26. Calendar year study, deaths by calendar year, migration on birthdays
Figure 27. Calendar year study, deaths by calendar year, migration at year-ends

7. LIFE INSURANCE AND ANNUITIES

7.1 Deterministic point of view

Let \( k_{x-1} q_x \) be the probability that a life \( (x) \) aged \( x \) will die between ages \( x + k - 1 \) and \( x + k \). Let \( k_{x} p_x \) be the probability that \( (x) \) will survive to age \( x + k \). Let \( i \) be the nominal interest rate and let \( v = 1/(1 + i) \).

Let \( A_{x:n}^1 \) denote an \( n \)-year term insurance of 1 payable at the end of the year of death. Then

\[
A_{x:n}^1 = \sum_{k=0}^{n-1} v^{k+1} q_x .
\]

Eq. 22

Let \( A_{x:n}^1 \) denote an \( n \)-year pure endowment of 1 payable at the end of the \( n \)th year when \( (x) \) survives. Then

\[
A_{x:n}^1 = v^a n p_x .
\]

Eq. 23
Let $A_{x:n}$ denote an n-year endowment insurance of 1 payable either at the end of the year of death or at the end of the nth year when $(x)$ survives. Then

$$A_{x:n} = A_{x}^{1} + A_{x:n}^{1}. \quad \text{Eq. 24}$$

Let $a_{x:n}$ denote an n-year annuity of 1 payable at the end of each year while $(x)$ survives. Then it is called an annuity immediate and

$$a_{x:n} = \sum_{k=1}^{n} v^{k} p_{x}. \quad \text{Eq. 25}$$

Let $\ddot{a}_{x:n}$ denote an n-year annuity of 1 payable at the beginning of each year while $(x)$ survives. Then it is called an annuity due and

$$\ddot{a}_{x:n} = \sum_{k=0}^{n-1} v^{k} p_{x} = 1 + a_{x:n-1}. \quad \text{Eq. 25}$$

An annuity due can also be interpreted as an endowment insurance with $\ddot{a}_{k}$ payable at the year of death and $\ddot{a}_{n}$ payable at the date of maturity. Hence

$$\ddot{a}_{x:n} = \sum_{k=0}^{n-1} \ddot{a}_{k+1} q_{x} + \ddot{a}_{n} p_{x}. \quad \text{Eq. 25}$$

Therefore, we can consider an n-year term life contingency function $\alpha_{x:n}^{1}$ with the death benefit $\alpha_{k}$ payable at the end of the year of death. Then

$$\alpha_{x:n}^{1} = \sum_{k=0}^{n-1} \alpha_{k+1} q_{x} \quad \text{Eq. 26}$$

and an n-year endowment contingency function with $\alpha_{n}$ payable at the date of maturity is

$$\alpha_{x:n} = \alpha_{x:n}^{1} + \alpha_{x:n}^{1},$$
where $\alpha_{x:n}$ is an $n$-year pure endowment of $\alpha_n$ at the date of maturity, namely

$$\alpha_{x:n}^1 = \alpha_{n:n}^1 p_x .$$

Eq. 27

By taking $\alpha_k = v^k$ in Eq. 26, we can obtain Eq. 22. In this case, Eq. 23 follows from Eq. 27.

These are the formulas for life insurance.

By taking $\alpha_k = \ddot{a}_k$ in Eq. 26, we can obtain

$$\ddot{a}^1_{x:n} = \sum_{k=0}^{n-1} \ddot{a}_{k+1} v^k q_x ;$$

$$\ddot{a}_{x:n} = \sum_{k=0}^{n-1} \ddot{a}_{k+1} v^k q_x + \ddot{a}_{n:n} p_x .$$

In this case, Eq. 25 follows from Eq. 27 and the above. These are the formulas for annuity due.

By taking $\alpha_k = a_k$ in Eq. 26, we can obtain

$$a^1_{x:n} = \sum_{k=0}^{n-1} a_{k+1} v^k q_x ;$$

$$a_{x:n} = \sum_{k=0}^{n-1} a_{k+1} v^k q_x + a_{n:n} p_x .$$

Finally, from Eqs. 24 and 25, we can derive

$$A_{x:n} + d\ddot{a}_{x:n} = \sum_{k=0}^{n-1} (v^{k+1} + d\ddot{a}_{k+1}) v^k q_x + (v^n + d\ddot{a}_{n:n}) p_x = 1 .$$

Likewise, we can obtain

$$A^1_{x:n} + d\ddot{a}^1_{x:n} = \sum_{k=0}^{n-1} v^k q_x = n q_x .$$
7.2 Stochastic point of view

Let \( K \) be the random variable of the integral future-life-time of \((x)\). Then its p.d.f.is

\[ k! q_x, 0 \leq k \leq \omega - x - 1, \text{ where } \omega \text{ is the terminal age.} \]

Let \( \alpha_{x,n} \) be an \( h \)-year deferred \( n \)-year life contingency function, with the random variable of the present value of the benefit being

\[
\alpha_K \quad K = h, h+1, h+2, \ldots, h+n-1
\]

and

\[
\alpha_{h+n} \quad K = h+n, h+n+1, h+n+2, \ldots
\]

Then

\[
\alpha_{x,n} = E[Z_{\alpha_{x,n}}] = \sum_{k=h}^{h+n-1} \alpha_{k+h} q_x + \alpha_{h+n} p_x. \tag{Eq. 28}
\]

As in Eqs. 26 and 27, we write

\[
\alpha_{x,n}^1 = \alpha_{h+n} p_x
\]

and

\[
\alpha_{x,n}^1 = \sum_{k=h}^{h+n-1} \alpha_{k+h} q_x.
\]

Let \( \alpha_{x,n} \) be an \( h \)-year deferred \( n \)-year endowment payable at the end of the year. Then

\[
\alpha_k = v^k, \quad h+1 \leq k \leq h+n.
\]

From Eq. 28, we have

\[
\alpha_{x,n} = \sum_{k=h}^{h+n-1} v^{k+1} q_x + v^{h+n} p_x. \tag{Eq. 29}
\]
Let $A_{x:n}^1$ be an $h$-year deferred $n$-year term insurance payable at the end of the year. Then from Eq. 28, we have

$$h|A_{x:n}^1 = \sum_{k=h}^{n} v^{k+1} h|q_x,$$

Eq. 30

since $\alpha_{h:n} = 0$. From Eqs. 29 and 30, we have

$$h|A_{x:n} = h|A_{x:n}^1 + v^{h+n} p_x \equiv h|A_{x:n} + h|E_x = h|A_{x:n} + h|A_{x:n}^1.$$

Let $a_{x:n}$ be an $h$-year deferred $n$-year annuity payable at the end of the year. By taking $\alpha_k = a_k - a_h$ in Eq. 28, we have

$$h|a_{x:n} = \sum_{k=h+1}^{h+n} (a_k - a_h) h|q_x + (a_{h+n} - a_h) h|p_x = \sum_{k=h+1}^{h+n} a_k h|q_x - a_{h+1} h|p_x + a_{h+n} h|p_x,$$

Eq. 31

and

$$h|a_{x:n}^1 = \sum_{k=h+1}^{h+n} a_k h|q_x.$$

Since $a_k = \nu + \nu^2 + \nu^3 + \ldots + \nu^k$ and $k|q_x = k|p_x - k+1|p_x$, from Eq. 31 we have

$$h|a_{x:n} = \sum_{k=h+1}^{h+n} \nu^k h|p_x.$$

Let $\ddot{a}_{x:n}$ be an $h$-year deferred $n$-year annuity payable at the beginning of the year. By taking

$$\alpha_k = \ddot{a}_{k+1} - \ddot{a}_k,$$

we can, likewise, obtain

$$h|\ddot{a}_{x:n} = \sum_{k=h}^{h+n-1} \ddot{a}_{k+1} h|q_x - \ddot{a}_h h|p_x + \ddot{a}_{h+n} h|p_x = \sum_{k=h}^{h+n-1} \nu^k h|p_x,$$

$$h|\ddot{a}_{x:n}^1 = \sum_{k=h}^{h+n-1} \ddot{a}_{k+1} h|q_x.$$
Let $T$ be the random variable of the life-until-death of $(x)$. Then its p.d.f. is $p_x \mu_{s+t}$, where $\mu_{s+t}$ is the force of mortality at age $x+t$.

Let $\alpha_{t+x}$ be an $h$-year deferred $n$-year continuous life contingency function with the present value of the death benefit at time $t$ being $\alpha_t$ and that of the maturity benefit being $\alpha_{h+n}$.

When $n = \omega - x$, $\alpha_{t+x}$ becomes $\alpha_x$, called an $h$-year deferred whole life continuous contingency function. When $h = 0$, they are denoted as $\alpha_{x+n}$ and $\alpha_x$, respectively.

Since the random variable of the present value of the benefit is

$$Z = \begin{cases} \alpha_t & h \leq T \leq h+n \\ \alpha_{h+n} & T > h+n \end{cases},$$

it follows that

$$E[Z_{t+x}] = \int_h^{h+n} \alpha_t \mu_{x+t} dt + \alpha_{h+n} p_x . \quad \text{Eq. 32}$$

By taking $\alpha_t = v^t$ in Eq. 32, we obtain

$$E[\alpha_{t+x}] = \int_h^{h+n} v^{t} \mu_{x+t} dt + v_{h+n}^{h+n} p_x . \quad \text{Eq. 33}$$

and

$$E[\alpha_{1+x}] = \int_h^{h+n} v^{t} \mu_{x+t} dt ,$$

where $\alpha_{t+x}$ (respectively, $\alpha_{1+x}$) is an $h$-year deferred $n$-year endowment (respectively, term) insurance of 1 payable at the moment of death or at the date of maturity.

Let $\alpha_{t+x}$ denote an $h$-year deferred $n$-year continuous annuity. Then
\[ Z = Z(\overline{a}_{x+n}) = \begin{cases} 0 & 0 \leq T \leq h \\ \overline{a}_r - \overline{a}_n & h \leq T \leq h + n \\ \overline{a}_{h+n} - \overline{a}_h & T > h + n \end{cases} \]

It follows from Eq. 32 that

\[ h\overline{a}_{x+n} = E[Z_{u_{x+n}}] = \int_h^{h+n} (\overline{a}_r - \overline{a}_n) p_x \mu_{x+t} dt + (\overline{a}_{h+n} - \overline{a}_h) _{h+n} p_x. \tag{Eq. 34} \]

Since

\[ \overline{a}_r = \frac{1 - \nu}{\delta}, \]

we can obtain from Eqs. 33 and 34 that

\[ h\overline{a}_{x+n} = (\overline{E}_x - h\overline{A}_{x+n}) / \delta = \overline{E}_x (1 - \overline{A}_{x+h+n}) / \delta, \]

where \( \overline{E}_x = \nu^h p_x \) and \( \delta \) is the force of interest.

### 7.3 Dynamic point of view

Let \( d_\delta(x,t) \) and \( d_\mu(x,t) \) be the discount function of interest and mortality, respectively.

Define an h-year deferred n-year continuous annuity as

\[ h\overline{a}_{x+n} = \int_h^{h+n} d_\delta(x,t) d_\mu(x,t) dt \]

and an h-year deferred n-year continuous term insurance as

\[ h\overline{A}_{x+n}^1 = \int_h^{h+n} d_\delta(x,t) \frac{d}{dt}[-d_\mu(x,t)] dt. \]

Then an h-year deferred n-year continuous endowment insurance is defined to be
where

\[ h+\delta \cdot E_{x+n} = h+\delta \cdot \tilde{A}_{x+n} \]

is an h-year deferred n-year pure endowment.

This is the model of life contingency functions based on discount functions. In particular, if we let

\[ d_\delta (x, t) = e^{-\delta t} \]

and

\[ d_\mu (x, t) = e^{-\int_0^t \mu_s \, ds} \]

then we can obtain the familiar (traditional) expressions for life contingency functions.

Another alternative is to let

\[ d_\delta (x, t) = e^{-\delta_x t} \]

where \( \delta_x \) is the force of interest at the issue age \( x \). By integration by parts, we can obtain

\[ h+\delta \cdot \tilde{A}_{x+n} = h+\delta \cdot \tilde{A}_{x+n} - d_\delta (x, h+n) d_\mu (x, h+n) \]

and

\[ h+\delta \cdot \tilde{A}_{x+n} = h+\delta \cdot \tilde{A}_{x+n} - \delta_x \cdot \tilde{A}_{x+n} \]

Note that \( \delta_x \) could be updated according to a certain national index. It can also include the expense factor for the calculation of the gross premiums, while \( d_\mu (x, t) \) could be updated according to the national life table. On the other hand, \( \tilde{a}(x, n) \) can always be approximated. As for the discrete case, the conventional approximations are handy.
8. NET ANNUAL PREMIUMS AND RESERVES

8.1 Net annual premiums

Let \( P_{h|n}^{x} \) be the continuously paid net level premium, or simply net premium of \( P_{h|n}^{x} \), with payments for \( r \) years.

In the annuity case, it only makes sense that \( r < h + n \). The reason is as follows. The insured pays \( r \) years of premiums when financially able, then starts receiving payments after \( h \) years for \( n \) years when financially needy. When \( h = 0 \), then the paying period should be less than the receiving period.

There is no such restriction in the insurance case. In fact, when \( h = 0 \), \( r \) is usually equal to \( n \). In this case, \( n \) \( P(A_{x}^{1}) \) is abbreviated as \( P_{x}^{1} \), \( n \) \( P(A_{x}^{1}) \) as \( P_{x}^{1} \) and \( n \) \( P(A_{x}^{1}) \) as \( P_{x}^{1} \).

Let \( L \) be the random variable of the present value of the insurer’s loss. Then

\[
L = Z(h|n^{x}) - rP_{h|n}^{x}Z(n^{x}).
\]

If \( E[L] = 0 \), then

\[
\frac{rP_{h|n}^{x}}{P_{x}^{1}} = \frac{\alpha^{x}}{a_{x|r}}.
\]

Hence we have

\[
\frac{rP_{h|n}^{x}}{P_{x}^{1}} = \frac{\alpha^{x}}{a_{x|r}},
\]
\[
\frac{rP_{h|n}^{x}}{P_{x}^{1}} = \frac{\alpha^{x}}{a_{x|r}},
\]
\[
\frac{rP_{h|n}^{x}}{P_{x}^{1}} = \frac{\alpha^{x}}{a_{x|r}},
\]
\[
\frac{rP_{h|n}^{x}}{P_{x}^{1}} = \frac{\alpha^{x}}{a_{x|r}}.
\]
and

\[ \overline{P}(h \mid A_{x+n}) = h A_{x+n} \div \overline{\alpha}_{x+n} \].

For the special cases, we have

\[ \overline{P}_{x+n} = \overline{A}_{x+n} \div \overline{\alpha}_{x+n} \],
\[ \overline{P}_{x+n}^1 = \overline{A}_{x+n}^1 \div \overline{\alpha}_{x+n} \]

and

\[ \overline{P}_x = \overline{A}_x \div \overline{\alpha}_x \].

For the discrete case, similar formulas can be derived.

8.2 Net premium reserves

We shall discuss the reserves based on the net level premium \( \overline{P}(h \mid \overline{\alpha}_{x+n}) \).

Define

\[ \overline{V}(h \mid \overline{\alpha}_{x+n}) = \text{the reserve needs to be provided for (x) by the insurer at the end of the t-th year, abbreviated as the reserve for (x) at the end of the t-th year or simply the reserve for (x + t)}. \]

Let \( U \) be the random variable of the future-life-time of (x + t). Then its p.d.f. is

\[ u P_{x+t \mid \mu_{x+t+u}} \].

Let \( L \) be the random variable of the loss of the insurer at the end of the t-th year. Then

\[ \overline{V}(h \mid \overline{\alpha}_{x+n}) = E[t \mid L], \]

the value of which is as follows.
i) If \( r < h \), then
\[
\frac{h-r}{t} \bar{\alpha}_{x+t} - \rho(h, \bar{\alpha}_{x}) \bar{\alpha}_{x+t-t} \quad t < r
\]
\[
\bar{\alpha}_{x+t-h} = \frac{h-r}{r} \bar{\alpha}_{x+t} \quad r \leq t < h
\]
\[
\bar{\alpha}_{x+t} + \gamma \quad h \leq t \leq h + n
\]

ii) If \( r = h \), then
\[
\frac{h-r}{t} \bar{\alpha}_{x+t} - \rho(h, \bar{\alpha}_{x}) \bar{\alpha}_{x+t-t} \quad t < h
\]
\[
\bar{\alpha}_{x+t-h} = \frac{h-r}{r} \bar{\alpha}_{x+t} \quad h \leq t \leq h + n
\]

iii) If \( r > h \), then
\[
\frac{h-r}{t} \bar{\alpha}_{x+t} - \rho(h, \bar{\alpha}_{x}) \bar{\alpha}_{x+t-t} \quad t < h
\]
\[
\bar{\alpha}_{x+t-h} = \frac{h-r}{r} \bar{\alpha}_{x+t} - \rho(h, \bar{\alpha}_{x}) \bar{\alpha}_{x+t-t} \quad h \leq t < r
\]
\[
\bar{\alpha}_{x+t} + \gamma \quad r \leq t \leq h + n
\]

The above formulas also hold for \( h, \bar{\alpha}_{x} \), with \( \bar{\alpha}_{x} \bar{\alpha}_{x} = 0 \).

If \( h = 0 \), then
\[
\bar{\alpha}_{x+t} = \frac{h-r}{t} \bar{\alpha}_{x+t} - \rho(h, \bar{\alpha}_{x}) \bar{\alpha}_{x+t-t} \quad t < r
\]
\[
\bar{\alpha}_{x+t} = \bar{\alpha}_{x+t} \quad r \leq t \leq n
\]

and
\[
\bar{\alpha}_{x+t} = \bar{\alpha}_{x+t} - \rho(h, \bar{\alpha}_{x}) \bar{\alpha}_{x+t-t} \quad t < r
\]
\[
\bar{\alpha}_{x+t} = \bar{\alpha}_{x+t} \quad r \leq t \leq n
\]
In the case of \( r = n \), we have

\[
\bar{\alpha}_{x+t:n-t} - \bar{P}(\bar{\alpha}_{x:n})\bar{a}_{x+t:n-t} \quad t < n
\]

and

\[
\bar{V}(\bar{\alpha}_{x:n}) = \bar{\alpha}_{x+n} \quad t = n
\]

We write \( \bar{V}(\bar{A}_{x:n}) \) and \( \bar{V}(\bar{A}^1_{x:n}) \), respectively as \( \bar{V}_{x:n} \) and \( \bar{V}^1_{x:n} \). Thus

\[
\bar{V}^1_{x:n} = \bar{V}(\bar{A}^1_{x:n}) = 1 \quad t = n
\]

9. VARYING LIFE CONTINGENCY FUNCTIONS

9.1 Increasing life contingency functions

Let \( (I\bar{\alpha})_{x:n} \) be an \( n \)-year continuous contingency function providing the present value of the death benefit \( (t+1)\bar{\alpha}_t \) at time \( t \) and the maturity benefit \( n\bar{\alpha}_x \), where \( x \) is the floor function of \( x \) (the greatest integer less than \( x \)). If the maturity benefit is 0, then the function is denoted by \( (I\bar{\alpha})^1_{x:n} \). Thus

\[
(I\bar{\alpha})_{x:n} = (I\bar{\alpha})^1_{x:n} + n\bar{\alpha}_{n,n} p_x .
\]

It follows from Eq. 31 that

\[
(I\bar{\alpha})^1_{x:n} = \int_0^n (t+1)\bar{\alpha}_t p_x \mu_{x+t} dt.
\]

Since the death benefit of both functions increases by 1 each year, they are called annually increasing life contingency functions with the difference only in the maturity benefit.
If the present value of the death benefit at time \( t \) is

\[
\frac{(tm + 1)}{m} \overline{\alpha}_t,
\]

then the above functions are denoted by \((I^{(m)}_{(m)} \overline{\alpha})_{x,n}\) and, called \( m \)thly increasing life contingency functions.

If the death benefit increases only for \( h \) years, then the pertinent functions are written as

\[(I^{(m)}_{(m)} \overline{\alpha})_{x,h}\]

and

\[(I^{(m)}_{(m)} \overline{\alpha})_{x,n}^{1}\].

9.2 Decreasing life contingency functions

Let \((D \overline{\alpha})_{x,n}^{1}\) be an \( n \)-year continuous contingency function providing the present value of the death benefit \((n - t)\overline{\alpha}_t\) at time \( t \). Then it follows from Eq. 32 that

\[
(D \overline{\alpha})_{x,n}^{1} = \int_{0}^{n} (n - t)\overline{\alpha}_t, p_x, \mu_x, dt.
\]

The death benefit decreases by 1 annually from \( n \) to 1. Thus such a function is called an annually decreasing life contingency function. Since the maturity benefit is 0, the notation \((D \overline{\alpha})_{x,n}\) is redundant.

If the present value of the death benefit at time \( t \) is

\[
(n - \frac{(tm)}{m})\overline{\alpha}_t,
\]

then the pertinent function is denoted by \((D^{(m)}_{(m)} \overline{\alpha})_{x,n}^{1}\), called \( m \)thly decreasing life contingency function. If the death benefit decreases only for \( h \) years, then the pertinent functions is written as \((D^{(m)}_{h} \overline{\alpha})_{x,n}^{1}\).
9.3 The supplementary relationships

Since

$$\frac{(tm+1)}{m} + (n - \frac{(tm)}{m}) = n + \frac{1}{m},$$

we have

$$(I^{(m)}\overline{\alpha})^{\text{\textdagger}}_{x,n} + (D^{(m)}\overline{\alpha})^{\text{\textdagger}}_{x,n} = (n + \frac{1}{m})\overline{\alpha}^{\text{\textdagger}}_{x,n}.$$  \hspace{1cm} \text{Eq. 35}

This supplementary relationship can also be seen from the following figure.

Figure 28. The supplementary relationship of $(D^{(m)}\overline{\alpha})^{\text{\textdagger}}_{x,n}$ and $(I^{(m)}\overline{\alpha})^{\text{\textdagger}}_{x,n}$

If $m \to \infty$, then from Eq. 35 we have

$$(\overline{\alpha})^{\text{\textdagger}}_{x,n} + (\overline{D}\overline{\alpha})^{\text{\textdagger}}_{x,n} = n\overline{\alpha}^{\text{\textdagger}}_{x,n},$$  \hspace{1cm} \text{Eq. 36}

where $(\overline{I}\overline{\alpha})^{\text{\textdagger}}_{x,n}$ is an n-year continually increasing life contingency function and $(\overline{D}\overline{\alpha})^{\text{\textdagger}}_{x,n}$ is an n-year continually decreasing life contingency function.

If the present value of the maturity benefit is $n\overline{\alpha}_{n}$, then the pertinent function is

$$(\overline{I}\overline{\alpha})_{x,n} = \lim_{m \to \infty} (I^{(m)}\overline{\alpha})_{x,n} = (\overline{I}\overline{\alpha})^{\text{\textdagger}}_{x,n} + n\overline{\alpha}_{n} p_{x},$$  \hspace{1cm} \text{Eq. 37}
The supplementary relationship in Eq. 36 can be seen from the following figure.

![Figure 29](image-url)

Figure 29. The supplementary relationship of \((D \bar{a})_{x,n}^1\) and \((I \bar{a})_{x,n}^1\)

The triangle representing \((I \bar{a})_{x,n}^1\) consists of all the horizontal line segments, each representing \(\alpha_{x,n-x}^1, 0 \leq s \leq n\). This relationship can also be proved as follows.

\[
(I \alpha)_{x,n}^1 = \int_0^n t \alpha_{I,n}^1 p_x \mu_{x+s} dt = \int_0^n \int_0^s ds \alpha_{I,n}^1 p_x \mu_{x+s} dt = \int_0^n \int_0^s \alpha_{I,n}^1 p_x \mu_{x+s} dt ds = \int_0^n \alpha_{x,n-x}^1 ds. \quad \text{Eq. 38}
\]

Similarly, we have

\[
(D \alpha)_{x,n}^1 = \int_0^n (n-t) \alpha_{D,n}^1 p_x \mu_{x+s} dt = \int_0^n \int_0^t ds \alpha_{D,n}^1 p_x \mu_{x+s} dt = \int_0^n \alpha_{x,n}^1 ds. \quad \text{Eq. 39}
\]

Combining Eqs. 38 and 39, we have

\[
(I \alpha)_{x,n} = \int_0^n (\alpha_{x,n-x}^1 + \alpha_{n,n}^1 p_x) ds = \int_0^n \alpha_{x,n-x}^1 ds. \quad \text{Eq. 40}
\]

On the other hand, we can use the integration by parts to obtain

\[
(I \alpha)_{x,n} = \int_0^n \alpha_{I,n}^1 p_x dt + \int_0^n t \alpha_{I,n}^1 p_x dt; \quad \text{Eq. 41}
\]

\[
(D \alpha)_{x,n} = n \alpha_{0} - \int_0^n \alpha_{D,n}^1 p_x dt + \int_0^n (n-t) \alpha_{I}^1 p_x dt. \quad \text{Eq. 42}
\]
By taking $\alpha_t = v^t$ in the hitherto derived functions, we can obtain the formulas for

$$(I\overline{A})_{x\infty}, \ (I\overline{A})_{x\infty}, \ (D\overline{A})_{x\infty}, \ (l^{(m)}\overline{A})_{x\infty}, \ (l^{(m)}\overline{A})_{x\infty}, \ (l^{(m)}\overline{A})_{x\infty}, \ (I_{h}^{(m)}\overline{A})_{x\infty}, \ (I_{h}^{(m)}\overline{A})_{x\infty},$$

$$(D_{h}^{(m)}\overline{A})_{x\infty}, \ (I\overline{A})_{x\infty}, \ (I\overline{A})_{x\infty} \text{ and } (D\overline{A})_{x\infty}.$$ 

By taking $\alpha'_{t} = v^t$, we can obtain the formulas for

$$(l\overline{a})_{x\infty}, \ (l\overline{a})_{x\infty}, \ (D\overline{a})_{x\infty}, \ (l^{(m)}\overline{a})_{x\infty}, \ (l^{(m)}\overline{a})_{x\infty}, \ (l^{(m)}\overline{a})_{x\infty}, \ (I_{h}^{(m)}\overline{a})_{x\infty}, \ (I_{h}^{(m)}\overline{a})_{x\infty}, \ (D_{h}^{(m)}\overline{a})_{x\infty},$$

$$(\overline{l\overline{a}})_{x\infty}, \ (\overline{l\overline{a}})_{x\infty} \text{ and } (\overline{D\overline{a}})_{x\infty}.$$ 

Note that, in the case of annuities, $\alpha_t = \overline{a}_t$ except for the last three types and for the last three types,

$$\overline{a}_t = \lim_{\Delta t \to 0} \int_{t}^{t+\Delta t} v^s ds = \lim_{\Delta t \to 0} (\overline{a}_{t+\Delta t} - \overline{a}_t) = 0.$$

Hence, from Eq. 37, we have

$$(I\overline{A})_{x\infty} = (I\overline{A})_{x\infty} + \overline{a}_{x\infty}$$

and

$$(\overline{l\overline{a}})_{x\infty} = (\overline{l\overline{a}})_{x\infty}.$$ 

From Eq. 36, we have

$$(I\overline{A})_{x\infty} + (D\overline{A})_{x\infty} = n\overline{A}_{x\infty}$$

and

$$(l\overline{a})_{x\infty} + (D\overline{a})_{x\infty} = n\overline{a}_{x\infty}.$$ \hspace{1cm} \text{Eq. 43}
Furthermore, from Eq. 38 and 39, we have

\[(\tilde{Ia})_{x,n} = (\tilde{Ia})^1_{x,n} = \int_0^n \tilde{a}^1_{x,n-s} ds\]

and

\[(\tilde{Da})^1_{x,n} = \int_0^n \tilde{a}^1_{x,n-s} ds .\]

Let \(\tilde{a}' = v'\). Since \(\tilde{a} = 0\), from Eq. 41 we have

\[(\tilde{Ia})_{x,n} = \int_0^n tv', p_x dt .\]  

Eq. 44

As a special case, we have

\[\tilde{a}_{x,n} = \int_0^n v' p_x dt .\]  

Eq. 45

From Eq. 42, we can also derive Eq. 43 as follows:

\[(\tilde{Da})^1_{x,n} = \int_0^n (n-t)v', p_x dt = n\tilde{a}_{x,n} - (\tilde{Ia})_{x,n} .\]

For insurance, from Eqs. 38-40, we can obtain

\[(\tilde{IA})^1_{x,n} = \int_0^n \tilde{A}^1_{x,n-s} ds ,\]

\[(\tilde{DA})^1_{x,n} = \int_0^n \tilde{A}^1_{x,n-s} ds \]

and

\[(\tilde{Ia})_{x,n} = \int_0^n \tilde{a}^1_{x,n-s} ds .\]  

Eq. 46

Let \(\tilde{a} = v'\). From Eqs. 41-43, we have

\[(\tilde{Ia})_{x,n} = \int_0^n v' , p_x dt - \int_0^n t\delta v' , p_x dt = \tilde{a}_{x,n} - \delta(\tilde{Ia})_{x,n} .\]

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From Eqs. 42, 45 and 46 we have

\[
(\overline{D} \overline{A})_{x:n} = n - \int_0^n (n-t)v' \, p_x \, dt - \int_0^n v' \, p_x \, dt = n - \delta(\overline{D} \overline{a})_{x:n} - \overline{a}_{x:n} = n - \overline{a}_{x:n} + \delta \overline{a}_{x:n} + \delta(\overline{I} \overline{a})_{x:n}.
\]

Similar to Eq. 40, we can obtain

\[
(\overline{I} \overline{a})_{x:n} = \sum_{k=0}^{n-1} k! \overline{a}_{x:n-k}
\]

and then

\[
(\overline{I} \overline{A})_{x:n} + \delta(\overline{I} \overline{a})_{x:n} = \sum_{k=0}^{n-1} (k! \overline{A}_{x:n-k} + \delta k! \overline{a}_{x:n-k}) = \sum_{k=0}^{n-1} k \, E_x = \overline{a}_{x:n}.
\]

10. DERIVATIVES OF LIFE CONTINGENCY FUNCTIONS

10.1 Derivatives of continuous life insurance and annuities

Let \( l_x \) be the survivorship function. Since

\[
\frac{dl_x}{dx} = -l_x \mu_x,
\]

we have

\[
\frac{dp_x}{dx} = \frac{d \left( \frac{l_{x+t}}{l_x} \right)}{dx} = \frac{l_{x+t} \mu_{x+t} - l_x \mu_x}{l_x} = \frac{l_{x+t} (-l_x \mu_x)}{l_x^2} = p_x (\mu_x - \mu_{x+t}). \tag{Eq. 47}
\]

It follows that

\[
\frac{dE_x}{dx} = \frac{d(v' \, p_x)}{dx} = v' \, p_x (\mu_x - \mu_{x+t}) = E_x (\mu_x - \mu_{x+t}). \tag{Eq. 48}
\]

Using Eq. 47, we can differentiate Eq. 45 to obtain

\[
\frac{d\overline{I} \overline{a}_{x:n}}{dx} = \int_0^{l_{x+n}} \frac{d}{dx} \left( p_x (\mu_{x+t}) \right) \, dt + \overline{a}_{x+n \, h+n} \, p_x (\mu_x - \mu_{x+h+n}).
\]
Since
\[
\frac{d}{dx}(p_x \mu_{x+t}) = \frac{d(l_{x+t} \mu_{x+t})}{dx} = \left( \frac{Y_x}{l_x} \right) \frac{d(l_{x+t} \mu_{x+t})}{dx} + p_x \mu_{x+t} \mu_x,
\]

it follows that
\[
\frac{d}{dx}(\bar{\alpha}_{x+n}) = \int_{h}^{h+n} \left( \frac{\bar{\alpha}_{x+t}}{l_x} \right) \frac{d(l_{x+t} \mu_{x+t})}{dx} \, dt + \mu_x \int_{h}^{h+n} \bar{\alpha}_{x+t} p_x \mu_{x+t} \, dt + \bar{\alpha}_{h+n \cdot h+n} p_x (\mu_x - \mu_{x+h+n}).
\]

Using the integration by parts, we have
\[
\frac{d}{dx}(\bar{\alpha}_{x+n}) = \mu_x \bar{\alpha}_{x+n} - \int_{h}^{h+n} \bar{\alpha}_{x+t} p_x \mu_{x+t} \, dt - \bar{\alpha}_{h \cdot h} p_x \mu_{x+h}.
\] Eq. 49

10.2 Derivatives of discrete life insurance and annuities

From Eqs. 28 and 47, we can derive
\[
\frac{d}{dx}(\bar{\alpha}_{x+n}) = \sum_{k=h}^{h+n-1} \bar{\alpha}_{k+1} p_x (\mu_x - \mu_{x+k}) - \sum_{k=h}^{h+n-1} \bar{\alpha}_{k} p_x (\mu_x - \mu_{x+k+1}) + \bar{\alpha}_{n+h \cdot h+n} p_x (\mu_x - \mu_{x+h+n})
\]

\[
= \sum_{k=h}^{h+n-1} [\alpha_{k+1} p_x (\mu_x - \mu_{x+k}) + \alpha_{k} p_x (\mu_x - \mu_{x+k+1})] + \bar{\alpha}_{n+h \cdot h+n} p_x (\mu_x - \mu_{x+h+n})
\]

\[
= \mu_x \bar{\alpha}_{x+n} + \sum_{k=h}^{h+n-1} (\alpha_{k} - \alpha_{k+1}) p_x \mu_{x+k}.
\] Eq. 50

Let \( \alpha_k = v^k \). Then
\[
\frac{d}{dx}(A_{x+n}) = \mu_x \bar{\alpha}_{x+n} + \sum_{k=h}^{h+n-1} (v^k - v^{k+1}) k p_x \mu_{x+k} = \mu_x \bar{\alpha}_{x+n} + \sum_{k=h}^{h+n-1} E_s \mu_{x+k},
\]

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where \( d = \frac{i}{1+i} \).

Let \( \alpha_k = \ddot{a}_k \). Then

\[
\frac{d \alpha_k}{dx} = \mu_x \dot{a}_{x,n} + \sum_{k=h}^{h+n-1} (\ddot{a}_k - \ddot{a}_{k+1}) p_x \mu_{x+k} \\
= \mu_x \dot{a}_{x,n} - \sum_{k=h}^{h+n-1} E_x \mu_{x+k}.
\]

Let \( \alpha_k = a_k \). Then

\[
\frac{d \alpha_k}{dx} = \mu_x \dot{a}_{x,n} + \sum_{k=h}^{h+n-1} (a_k - a_{k+1}) p_x \mu_{x+k} \\
= \mu_x \dot{a}_{x,n} - \sum_{k=h}^{h+n-1} E_x \mu_{x+k}.
\]

10.3 Derivatives of varying life insurance and annuities

From Eqs. 40 and 49, we can derive

\[
\frac{d(\overline{I} \alpha)}{dx} = \int_0^x \frac{d \overline{C}_{x,n-s}}{dx} ds \\
= \int_0^x \overline{C}_{x,n-s} \mu_x ds - \int_0^x \overline{C}_{x,n-s} p_x \mu_{x+s} dt ds - \int_0^x \overline{C}_{x,n-s} p_x \mu_{x+s} ds \\
= \mu_x (\overline{I} \alpha) - \int_0^x \overline{C}_{x,n-s} p_x \mu_{x+s} dt - \int_0^x \overline{C}_{x,n-s} p_x \mu_{x+s} ds. \tag{Eq. 51}
\]
11. SOME USEFUL THEOREMS IN ACTUARIAL MATHEMATICS

Theorem A. Let a, c, d and e be positive numbers. Then the function

\[ f(x) = \frac{dx + e}{\sqrt{ax^2 + c}} \]

attains its maximum value

\[ \sqrt{\frac{d^2}{a} + \frac{e^2}{c}} \text{ at } x = \frac{cd}{ae}. \]

Proof. We first derive

\[ f'(x) = \frac{d\sqrt{ax^2 + c} - \frac{ax(dx + e)}{\sqrt{ax^2 + c}}}{\sqrt{ax^2 + c}} = \frac{cd - aex}{\sqrt{(ax^2 + c)^3}}; \]

\[ f''(x) = \frac{-ae\sqrt{ax^2 + c} - \frac{3a(cd - aex)}{\sqrt{(ax^2 + c)^3}}}{\sqrt{(ax^2 + c)^3}} = -\frac{ae(ax^2 + c)^3 + 3a(cd - aex)}{\sqrt{(ax^2 + c)^5}}. \]

Since the value of \( f''(x) \) at the critical point \( x = \frac{cd}{ae} \) is

\[ ae \left[ a \left( \frac{cd}{ae} \right)^2 + c \right]^3 \]

\[ - \frac{e \left[ a \left( \frac{cd}{ae} \right)^2 + c \right]^5}, \]

the maximum value is

\[ f \left( \frac{cd}{ae} \right) = \frac{a \left( \frac{cd}{ae} \right) + e}{\sqrt{a \left( \frac{cd}{ae} \right)^2 + c}} = \frac{c \left( \frac{d^2}{a} + \frac{e^2}{c} \right)}{e \left( \frac{d^2}{a} + \frac{e^2}{c} \right)} = \sqrt{\frac{d^2}{a} + \frac{e^2}{c}}. \]
Corollary A. For an insurance organization, let $S$ denote the random loss on a segment of
its risks and let $x$ be the retention limit the minimizes the probability

$$
\Pr \left( \frac{S - E[S]}{\sqrt{\text{Var}[S]}} > f(x) \right),
$$

where $f(x)$ is the ratio of the security loading $g(x) = dx + e$ and the standard deviation

$$
h(x) = \sqrt{\text{Var}[S]} = \sqrt{ax^2 + c}.
$$

Then $x = \frac{cd}{ae}$ and

$$
f \left( \frac{cd}{ae} \right) = \sqrt{\frac{d^2}{a} + \frac{e^2}{c}}.
$$

Corollary B. Let $a$, $b$, $c$, $d$ and $e$ be positive numbers such that $4ac > b^2$ and $2ae > bd$. Then

$$
f(x) = \frac{dx + e}{\sqrt{ax^2 + bx + c}}
$$

attains its maximum value

$$
\sqrt{\frac{d^2}{a} + (2ae - bd)^2}
\sqrt{a(4ac - b^2)}
$$

at $x = \frac{2cd - be}{2ae - bd}$.

Proof. Write

$$
f(x) = \frac{d \left( x + \frac{b}{2a} \right) + \frac{2ae - bd}{2a}}{\sqrt{a \left( x + \frac{b}{2a} \right)^2 + \frac{4ac - b^2}{4a}}}
$$

and use Theorem A.
Theorem B. Let \( f(x) = qb\exp(-bx) \) and \( g(x) = -\exp(-ax) \). Then

\[
h(d;c) = \int_{0}^{\infty} f(x)g(d - cx)dx = -\frac{qb\exp(-ad)}{b - ca}.
\]

Corollary C. Let \( p \) be the probability that a property will not be damaged in the next period and let \( f(x) \) in Theorem B be the probability density function of a positive random variable \( X \) with \( q = 1 - p \). If the owner of the property with wealth \( w \) has a utility function \( g(x) \) in Theorem B and is offered an insurance policy that will pay \( 1 - c \) portion of any loss during the next period, then the maximum premium \( G \) that the property owner will pay for this insurance is

\[
G = \frac{1}{a} \ln \frac{p + \frac{qb}{b - a}}{p + \frac{qb}{b - ca}}.
\]

Proof. Equating the utilities with and without insurance, we have

\[
p g(w - G) + h(w - G; c) = p g(w) + h(w; 1).
\]

It follows from Theorem B that

\[
-p\{\exp[-a(w - G)]\} - \frac{qb}{b - a}\exp[-a(w - G)]
\]

\[
= -p\exp(-aw) - \frac{qb}{b - a}\exp(-aw)
\]

so that

\[
\left(p + \frac{qb}{b - ca}\right)\exp(aG) = p + \frac{qb}{b - a}.
\]

The corollary follows.
Theorem C. Let
\[ f(x) = \frac{2}{a} \left(1 - \frac{x}{a}\right), \quad 0 \leq x \leq a, \]
be the probability density function of a random variable X. Then
\[ E[X^n] = \frac{a^n}{\binom{n+2}{2}}. \]

Corollary D. The mean and variance of the random variable X in Theorem C are \( \frac{a}{3} \) and \( \frac{a^2}{18} \), respectively.

Theorem D. A decision maker has wealth w, has a utility function
\[ u(x) = x^r, \quad 0 < x < 1 \]
and faces a random loss X with a uniform distribution on [0,w]. Then the maximum amount this decision maker will pay for the complete insurance against the random loss is
\[ G = \left[1 - \left(\frac{1}{r+1}\right)^{\frac{1}{r}}\right]w. \]

Proof. Equating the utilities with and without insurance, we have
\[ (w - G)^r = \int_0^w \frac{1}{w} (w - x)^r \, dx. \]

It follows that
\[ (w - G)^r = \frac{w^r}{r+1}. \]

The theorem follows.
Theorem E. Assume that a decision maker will retain wealth \( w \) with probability \( p \) and will suffer a loss \( c \) with probability \( q = 1 - p \). Based on the utility function 

\[
 u(x) = x - ax^2, \quad 0 < x < \frac{1}{2a} \quad (a > 0),
\]

the maximum insurance premium that the decision maker will pay for the complete insurance is 

\[
 G = w - \frac{1}{2a} \left\{ 1 - \left[ 1 - 4apw(1 - aw) + 4aq(w - c) - 4a^2q(w - c)^2 \right]^{\frac{1}{2}} \right\}.
\]

**Proof.** Equating the utilities with and without insurance, we have 

\[
 (w - G) - a(w - G)^2 = pw(1 - aw) + q(w - c)[1 - a(w - c)].
\]

It follows that 

\[
 w - G = \frac{1}{2a} \left\{ 1 - \left[ 4apw(1 - aw) + 4aq(w - c) - 4a^2q(w - c)^2 \right]^{\frac{1}{2}} \right\}.
\]

The theorem follows.

Theorem F. Let \( X_i, i = 1, 2, 3, \ldots, n \), be nonnegative mutually independent random variable with the probability density function \( f_i(t) \). If the moment generating function \( M_{X_i}(t) \) of each \( X_i \) is finite on some interval, then the convolution \( f_1 * f_2(x) \) is the unique probability density function of \( S = \sum_{i=1}^{n} X_i \).

**Proof.** We shall only prove the case with \( n = 2 \). For any \( t \) in the given interval, we have 

\[
 M_S(t) = \int_0^\infty e^{ty} \int_0^t f_1(x - y)f_2(y)dydx = \int_0^\infty e^{yv} f_2(v) \int_0^\infty e^{-zv} f_1(z)dz dy,
\]

where \( z = x - y \). Hence \( M_S(t) = M_{X_1}(t)M_{X_2}(t) \) and hence the theorem follows.
GLOSSARY

**Balducci assumption:** When \(0 < t < 1\), the mortality rate \(q_x\) cannot be found in a life table.

Under this assumption, the reciprocal interpolation is used.

**U-assumption:** When \(0 < t < 1\), the mortality rate \(q_x\) cannot be found in a life table. Under this assumption, the linear interpolation is used.

**CSO:** The acronym for Commissioners Standard Ordinary.

**SOA:** The acronym for Society of Actuaries.

**ARCH:** The acronym for Actuarial Research Clearing House, which is one of the two SOA publications of articles, the other is Transactions.

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The acronym STEM stands for “science, technology, engineering and mathematics”. In accordance with the National Science Teachers Association (NSTA), “A common definition of STEM education is an interdisciplinary approach to learning where rigorous academic concepts are coupled with real-world lessons as students apply science, technology, engineering, and mathematics in contexts that make connections between school, community, work, and the global enterprise enabling the development of STEM literacy and with it the ability to compete in the new economy”. The problem of this country has been pointed out by the US Department of Education that “All young people should be prepared to think deeply and to think well so that they have the chance to become the innovators, educators, researchers, and leaders who can solve the most pressing challenges facing our nation and our world, both today and tomorrow. But, right now, not enough of our youth have access to quality STEM learning opportunities and too few students see these disciplines as springboards for their careers.” STEM learning and applications are very popular topics at present, and STEM related careers are in great demand. According to the US Department of Education reports that the number of STEM jobs in the United States will grow by 14% from 2010 to 2020, which is much faster than the national average of 5-8 % across all job sectors. Computer programming and IT jobs top the list of the hardest to fill jobs. Despite this, the most popular college majors are business, law, etc., not STEM related. For this reason, the US government has just extended a provision allowing foreign students that are earning degrees in STEM fields a seven month visa extension, now allowing them to stay for up to three years of “on the job training”. So, at present STEM is a legal term.
The acronym STEAM stands for “science, technology, engineering, arts and mathematics”. As one can see, STEAM (adds “arts”) is simply a variation of STEM. The word of “arts” means application, creation, ingenuity, and integration, for enhancing STEM inside, or exploring of STEM outside. It may also mean that the word of “arts” connects all of the humanities through an idea that a person is looking for a solution to a very specific problem which comes out of the original inquiry process. STEAM is an academic term in the field of education. The University of San Diego and Concordia University offer a college degree with a STEAM focus. Basically STEAM is a framework for teaching or R&D, which is customizable and functional, thence the “fun” in functional. As a typical example, if STEM represents a normal cell phone communication tower looking like a steel truss or concrete column, STEAM will be an artificial green tree with all devices hided, but still with all cell phone communication functions. This ebook series presents the recent evolutionary progress in STEAM with many innovative chapters contributed by academic and professional experts.

This ebook chapter, “EVOLUTIONARY MATHEMATICS AND SCIENCE FOR LIFE CONTINGENCY INVESTIGATION” is Dr. Hung-ping Tsao’s collection of thoughts, works and articles about various basic life contingency problems encountered through his eight years of actuarial career plus seventeen years of teaching. From time to time, he would share his innovative ideas with all levels of audience by giving talks to college students in U.S. as well as in Taiwan. He even went to Shanghai, China in 1996 to help develop the actuarial education.